Foreword: Mathematical tools are often used to obtain qualitative and quantitative information of the evolution (dynamics) of physical systems. These tools are of utmost importance in studies on applied mathematics, science and engineering. In this workshop, we discuss how different mathematical tools, typically studied in several college level math courses, are related and used to understand physical systems. We use simple but illustrative examples, built from physically motivated (word) questions, and introduce necessary concepts that are new but useful in our approach. We hope that this workshop will serve as a proxy for you to understand the various concepts discussed in college level math courses, and help you establish abstract understanding of mathematics on applied problems. This workshop is no substitute for the actual math courses, but is the starting point to understand the connections. Please ask questions if you feel puzzled!

Some of the material in this workshop can be found in “Nonlinear Dynamics And Chaos: With Applications To Physics, Biology, Chemistry, And Engineering”, Steven Strogatz, Westview Press, 2001
May 21st
Lecture 1: Review of important concepts
Continuous functions

- Take the infinitesimal limit

\[
\frac{df}{dx} = f' = \lim_{\Delta x \to 0} \frac{\Delta f}{\Delta x}
\]

If \( f(t) \), \( \frac{df}{dt} = \dot{f} \) denotes the instantaneous change of \( f \) over time.

Link to real world: in many problems you can only describe \( \dot{f}, \ddot{f}, f', f'' \), etc.
Continuous functions

- Take the infinitesimal limit

\[ \frac{df}{dx} = f' = \lim_{\Delta x \to 0} \frac{\Delta f}{\Delta x} \]

- Slope: rate of change of the function \( df/dx = f' = \lim_{\Delta x \to 0} \Delta f/\Delta x \)
Continuous functions

- Take the infinitesimal limit

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\frac{df}{dx} = f' = \lim_{\Delta x \to 0} \frac{\Delta f}{\Delta x}
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- If \( f(t) \), \( \frac{df}{dt} = \dot{f} \) denotes the instantaneous change of \( f \) over time
- Link to real world: in many problems you can only describe \( \dot{f}, \ddot{f}, f', f'' \), etc
Calculus Example

- Consider the Newton’s law of motion

\[ \frac{dx}{dt} = \dot{x} = v, \]
\[ \frac{dv}{dt} = \dot{v} = a, \]
\[ F = ma = m\ddot{x}. \]

Finding \( F \) is trivial. Given \( F \), finding \( a \), then \( v \), then \( x \), can be difficult.

Message: We can link quantities of physical interest to mathematical equations. We’ll discuss how to solve for a subset of these equations later.
Calculus Example

- Consider the Newton’s law of motion

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\frac{dx(t)}{dt} = \dot{x} = v, \quad \frac{dv(t)}{dt} = \dot{v} = a, \quad F = ma = m\ddot{x}
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- Finding $F$? trivial. Given $F$, find $a$, then $v$, then $x$, can be difficult

- Message: We can link quantities of physical interest to mathematical equations
Calculus Example

- Consider the Newton’s law of motion

\[ \frac{dx(t)}{dt} = \dot{x} = \nu, \quad \frac{dv(t)}{dt} = \dot{\nu} = a, \quad F = ma = m\ddot{x} \]

- Finding \( F \) is trivial. Given \( F \), find \( a \), then \( \nu \), then \( x \), can be difficult

- Message: We can link quantities of physical interest to mathematical equations

- We’ll discuss how to solve for a subset of these equations later
Taylor series expansion

- Instantaneous change of a function — focus on ‘local’ behavior
Taylor series expansion

- Instantaneous change of a function — focus on ‘local’ behavior
- Taylor series expansion gives local information to certain order

Expanding a function $f$ near a point $x_0$ or $(x_0, y_0)$:

$\begin{align*}
f(x) &\approx f(x_0) + f'(x_0)(x-x_0) + \frac{1}{2}f''(x_0)(x-x_0)^2 + \cdots \\
f(x, y) &\approx f(x_0, y_0) + f_x(x_0, y_0)(x-x_0) + f_y(x_0, y_0)(y-y_0) + \frac{1}{2}f_{xx}(x_0, y_0)(x-x_0)^2 + f_{xy}(x_0, y_0)(x-x_0)(y-y_0) + \frac{1}{2}f_{yy}(x_0, y_0)(y-y_0)^2 + \cdots
\end{align*}$

Local approximation as plane, quadratic surface, etc.

Are the following functions differentiable? Where do you see trouble?
Taylor series expansion

- Instantaneous change of a function — focus on ‘local’ behavior
- Taylor series expansion gives local information to certain order
- Expanding a function \( f \) near a point \( x_0 \) or \((x_0, y_0)\)

\[
f(x) \approx f(x_0) + f'(x)|_{x=x_0}(x - x_0) + \frac{1}{2} f''(x)|_{x=x_0}(x - x_0)^2 + \cdots
\]

\[
f(x, y) \approx f(x_0, y_0) + f_x(x)|_{x=x_0, y=y_0}(x - x_0) + f_y(y)|_{x=x_0, y=y_0}(y - y_0)
+ \frac{1}{2} [f_{xx}(x)|_{x=x_0, y=y_0}(x - x_0)^2 + f_{yy}(y)|_{x=x_0, y=y_0}(y - y_0)^2
+ 2f_{xy}(x)|_{x=x_0, y=y_0}(x - x_0)(y - y_0)] + \cdots
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$$f(x) \approx f(x_0) + f'|_{x=x_0} (x - x_0) + \frac{1}{2} f''|_{x=x_0} (x - x_0)^2 + \cdots$$

$$f(x, y) \approx f(x_0, y_0) + f_x|_{x=x_0, y=y_0} (x - x_0) + f_y|_{x=x_0, y=y_0} (y - y_0)$$

$$+ \frac{1}{2} [f_{xx}|_{x=x_0, y=y_0} (x - x_0)^2 + f_{yy}|_{x=x_0, y=y_0} (y - y_0)^2$$

$$+ 2f_{xy}|_{x=x_0, y=y_0} (x - x_0)(y - y_0)] + \cdots$$

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- Are the following functions differentiable? Where do you see trouble?
Euler’s formula

- Euler’s formula is given as

\[ e^{ix} = \cos(x) + i\sin(x) \]
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- This comes from exponentiating a purely complex number

\[
e^{ix} = 1 + ix + \frac{(ix)^2}{2!} + \frac{(ix)^3}{3!} + \cdots + \frac{(ix)^n}{n!} + \cdots
\]

\[= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} \cdots + i \left( x - \frac{x^3}{3!} + \frac{x^5}{5!} \cdots \right) = \cos(x) + i \sin(x) \]
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  \]

- Hence we can also write
  \[
  \cos(x) = \frac{e^{ix} + e^{-ix}}{2}, \quad \sin(x) = \frac{e^{ix} - e^{-ix}}{2i}
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- Why important? Often analyze oscillatory systems (season, day-night)
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- Why important? Often analyze oscillatory systems (season, day-night)

- Extension

\[
\cosh(x) = \frac{e^x + e^{-x}}{2}, \quad \sinh(x) = \frac{e^x - e^{-x}}{2}
\]
Differential equations

- We already see one example of differential equation $m\ddot{x} = F(x, t)$
Differential equations

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- For unknown function $x$, usually in the form

$$F(x, x', x'', \cdots, t) = 0$$

Example:

$$x' = t, \quad x'' = x + t, \quad \ddots$$

Can be VERY difficult with nonlinearity and with partial derivatives.
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- Can be VERY difficult with nonlinearity and with partial derivatives
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- Focus on what each term means (so you feel how to construct a model and interpret a solution)
Matrix Algebra

- We can usually write a system of equations into matrix form

\[
\begin{pmatrix}
a_{11} & a_{12} \\
a_{21} & a_{22}
\end{pmatrix}
\begin{pmatrix}
x \\
y
\end{pmatrix} =
\begin{pmatrix}
b_1 \\
b_2
\end{pmatrix}
\text{ or }
\begin{pmatrix}
\dot{x} \\
\dot{y}
\end{pmatrix} =
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- Example: intersection of lines / 2D vector fields

- Properties of matrix leads to understandings of the system
- We’ll learn some important concept in matrix (linear) algebra
Numerical methods

- Without knowing analytic solutions, we can solve for differential equations numerically.

**Pseudo code:** (specify \(dx\), say, 0.1)

\[
x(0) = 1, \quad \dot{x}(0) = x(0) = 1,
\]

\[
x(dx) = x(0) + dx \times \dot{x}(0) = 1.
\]

\[
x[dx] = x(n-1), \quad \dot{x}(n-1) = x(n-1), \quad x(ndx) = x(n-1) + dx \times x(n-1).
\]

Some caveat: accuracy, stability.
Numerical methods

- Without knowing analytic solutions, we can solve for differential equations numerically
- Solving for $\dot{x} = x, x(0) = 1$
Numerical methods

- Without knowing analytic solutions, we can solve for differential equations numerically.
- Solving for $\dot{x} = x$, $x(0) = 1$
- Idea: Starting from $x(0) = 1$, estimate the slope $\dot{x}$, then advance ...
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$$x(0) = 1, \dot{x}(0) = x(0) = 1, x(dx) = x(0) + dx \times \dot{x}(0) = 1.1$$

$$x(dx) = 1.1, \dot{x}(dx) = 1.1, x(2dx) = x(dx) + dx \times \dot{x}(dx) = 1.21$$

$$\ldots$$

$$x[(n - 1)dx] = x_{n-1}, \dot{x}(n - 1) = x_{n-1}, x(ndx) = x_{n-1} + dx \times x_{n-1}$$

$$\ldots$$
Numerical methods

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- Solving for $\dot{x} = x$, $x(0) = 1$
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\[\vdots\]

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May 21st
Lecture 2: Examples in 1-D
Fishery in Tempe Town Lake

Consider fish population in TTL. Suppose there’s no fishing, food resource is abundant, no lag in maturation, what can we write down as an equation to describe the change in fish population?

The logistic equation

\[
\dot{f} = kf \left(1 - \frac{f}{C}\right),
\]

where \(k\) is the growth rate at no competition, \(C\) the carrying capacity indicating availability of resources.

Qualitative understanding: Is there anything you can say about the population without solving for the differential equations?
Fishery in Tempe Town Lake

- Consider fish population in TTL. Suppose there’s no fishing, food resource is abundant, no lag in maturation, what can we write down as an equation to describe the change in fish population?
- Now consider there is limitation of food resource, so growth is actually slowing down due to increasing population, what equation can we write down?

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- Now consider there is limitation of food resource, so growth is actually slowing down due to increasing population, what equation can we write down?

- The logistic equation
  \[
  \frac{df}{dt} = kf\left(1 - \frac{f}{C}\right),
  \]
  where \( k \) is the growth rate at no competition, \( C \) the carrying capacity indicating availability of resources

- Qualitative understanding: Is there anything you can say about the population without solving for the differential equations?
Quantitative analysis (ODE)

- Let’s consider the simpler part $df/dt = kf$. Physical meaning?

Separation of variables

$$\frac{df}{f} = kdt$$

$$\int \frac{df}{f} = \int kdt + c_0$$

$$\ln |f| = kt + c_0$$

$$f = e^{c_0} e^{kt}$$

The actual logistic equation?

Separation of variable, partial fraction, integrate

$$\int \left[ \frac{1}{f} + \frac{1}{c_0 - f} \right] df = kt + c_0$$

$$f = \frac{Cc_0}{1 + c_0 e^{kt}}$$
Quantitative analysis (ODE)

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f = e^{c_0}e^{kt}
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- Separation of variable, partial fraction, integrate

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\int \left[ \frac{1}{f} + \frac{1}{C-f} \right] df = kt + c_0,
\]

\[
f = \frac{Cc_0e^{kt}}{1 + c_0e^{kt}}
\]
Solution behavior

- Solution of the logistic equation subject to different initial conditions:
Solution behavior

- Solution of the logistic equation subject to different initial conditions:

- Qualitative understanding: Is there anything you can say about the population without solving for the differential equations?
Qualitative analysis (Dynamical System)

- How to describe behavior?
Qualitative analysis (Dynamical System)

- How to describe behavior?
- Phase portrait
Qualitative analysis (Dynamical System)

- How to describe behavior?
- Phase portrait

- Long term behavior? Fixed points? Stability?
Consider fishery

- How shall we control fishery?
Consider fishery

- How shall we control fishery?
- Concept: maintain stable population
Consider fishery

- How shall we control fishery?
- Concept: maintain stable population

Interpret each curve
Let’s try solve the example equations with numerical methods
Synchronization

- Firefly sync:

\[
\begin{align*}
\frac{d\Theta}{dt} &= \Omega + K_1 \sin(\theta - \Theta) \\
\frac{d\theta}{dt} &= \omega + K_2 \sin(\Theta - \theta)
\end{align*}
\]

Subtract the first equation from the second by writing \( \phi = \Theta - \theta \):

\[
\frac{d\phi}{dt} = \Omega - \omega - (K_1 + K_2) \sin \phi
\]

If \(|(\Omega - \omega)/(K_1 + K_2)| \leq 1\), then \( \phi^\star = \arcsin((\Omega - \omega)/(K_1 + K_2)) \) is an equilibrium point.

Two equilibrium points, one stable, corresponding to phase lock.
Synchronization

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\]

- If \( |(\Omega - \omega)/(K_1 + K_2)| \leq 1 \), \( \phi^* = \arcsin\left(\frac{\Omega - \omega}{K_1 + K_2}\right) \)
Synchronization

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\frac{d\phi}{dt} = \Omega - \omega - (K_1 + K_2) \sin \phi
\]

- If \(|(\Omega - \omega)/(K_1 + K_2)| \leq 1\), \(\phi^* = \arcsin\left(\frac{\Omega - \omega}{K_1 + K_2}\right)\)

- Two equilibrium points, one stable, corresponding to phase lock
Exercise for the day

• Suppose the population equation for fish in TTL is

\[ \dot{f} = f^2 - f^4, \]

how should we regulate fishery?

Hint: look for maximum of \( \dot{f} = f^2 - f^4 - p \), when the function reaches max at \( \dot{f} = 0 \), stable structure with max profit
Exercise for the day

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- Use numerical methods to solve for the above equation, with several choices of fishery control \( p \).

Hint: you should see that at maximum fishing rate solution converge from above fixed point and diverge from below.
Exercise for the day

- Suppose the population equation for fish in TTL is
  \[ \dot{f} = f^2 - f^4, \]
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- Use numerical methods to solve for the above equation, with several choices of fishery control \( p \)
  Hint: you should see that at maximum fishing rate solution converge from above fixed point and diverge from below

- The moth population dynamics in a forest obeys the logistic equation.
  With the introduction of one species of wood pecker, the moth decreases by the predation function \( p(f) = Af / (B + f) \).
  Discuss the dynamics.
  Hint: look for fixed points and discuss stability based on the touching of the curves
May 22\textsuperscript{nd}

Lecture 3: Linear 2D dynamics
Population dynamics

Let’s consider population migration

\[
\begin{pmatrix}
T^{n+1} \\
P^{n+1}
\end{pmatrix} = \begin{pmatrix} 0.3 & 0.8 \\ 0.7 & 0.2 \end{pmatrix} \begin{pmatrix} T^n \\
P^n
\end{pmatrix}
\]

with \(T^0 = 50000, P^0 = 50000\). Use the computer to check the long term behavior of \(T^{n+1}, P^{n+1}\).
Population dynamics

- Let’s consider population migration

\[
\begin{pmatrix}
T^{n+1} \\
\begin{array}{c}
p^{n+1}
\end{array}
\end{pmatrix} = \begin{pmatrix}
0.3 & 0.8 \\
0.7 & 0.2
\end{pmatrix}
\begin{pmatrix}
T^{n} \\
\begin{array}{c}
p^{n}
\end{array}
\end{pmatrix}
\]

with \( T^0 = 50000, P^0 = 50000 \). Use the computer to check the long term behavior of \( T^{n+1}, P^{n+1} \).

- There is a convergence to some \( T^*, P^* \).
Population dynamics

- Let’s consider population migration
  \[
  \begin{pmatrix}
  T^{n+1} \\
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  \end{pmatrix} = \begin{pmatrix} 0.3 & 0.8 \\ 0.7 & 0.2 \end{pmatrix} \begin{pmatrix} T^n \\ P^n \end{pmatrix}
  \]

  with \( T^0 = 50000, P^0 = 50000 \). Use the computer to check the long term behavior of \( T^{n+1}, P^{n+1} \).
- There is a convergence to some \( T^*, P^* \).
- The above equation then becomes
  \[
  \begin{pmatrix}
  T^* \\
  P^*
  \end{pmatrix} = \begin{pmatrix} 0.3 & 0.8 \\ 0.7 & 0.2 \end{pmatrix} \begin{pmatrix} T^* \\ P^* \end{pmatrix}
  \]

  But how to find them?
Eigenvalues and eigenvectors

Let $v^n = [T^n, P^n]^T$, $v^{n+1} = k[T^n, P^n]^T$ (because of alignment)

$$k \begin{pmatrix} T^n \\ P^n \end{pmatrix} = \begin{pmatrix} 0.3 & 0.8 \\ 0.7 & 0.2 \end{pmatrix} \begin{pmatrix} T^n \\ P^n \end{pmatrix}$$
Eigenvalues and eigenvectors

- Let $v^n = [T^n, P^n]^T$, $v^{n+1} = k[T^n, P^n]^T$ (because of alignment)
  
  $k \begin{pmatrix} T^n \\ P^n \end{pmatrix} = \begin{pmatrix} 0.3 & 0.8 \\ 0.7 & 0.2 \end{pmatrix} \begin{pmatrix} T^n \\ P^n \end{pmatrix}$

- So $(A - kl)v^n = [0, 0]^T$, or

  $\begin{pmatrix} 0.3 - k & 0.8 \\ 0.7 & 0.2 - k \end{pmatrix} \begin{pmatrix} T^n \\ P^n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

  But we need $[T^n, P^n]$ to be non-trivial
Eigenvalues and eigenvectors

- Let \( \mathbf{v}^n = [T^n, P^n]^T \), \( \mathbf{v}^{n+1} = k[T^n, P^n]^T \) (because of alignment)

\[
k \begin{pmatrix} T^n \\ P^n \end{pmatrix} = \begin{pmatrix} 0.3 & 0.8 \\ 0.7 & 0.2 \end{pmatrix} \begin{pmatrix} T^n \\ P^n \end{pmatrix}
\]

- So \((A - kI)\mathbf{v}^n = [0, 0]^T\), or

\[
\begin{pmatrix} 0.3 - k & 0.8 \\ 0.7 & 0.2 - k \end{pmatrix} \begin{pmatrix} T^n \\ P^n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}
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But we need \([T^n, P^n]\) to be non-trivial

- Hence \(|(A - kI)| = 0\) (explain determinant)
Eigenvalues and eigenvectors

- Let \( v^n = [T^n, P^n]^T, v^{n+1} = k[T^n, P^n]^T \) (because of alignment)

\[
k \begin{pmatrix} T^n \\ P^n \end{pmatrix} = \begin{pmatrix} 0.3 & 0.8 \\ 0.7 & 0.2 \end{pmatrix} \begin{pmatrix} T^n \\ P^n \end{pmatrix}
\]

- So \((A - kl)v^n = [0, 0]^T\), or

\[
\begin{pmatrix} 0.3 - k & 0.8 \\ 0.7 & 0.2 - k \end{pmatrix} \begin{pmatrix} T^n \\ P^n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}
\]

But we need \([T^n, P^n]\) to be non-trivial

- Hence \(|(A - kl)| = 0\) (explain determinant)

- For this case \(k_1 = 1, v_1 = [8, 7]^T, k_2 = -0.5, v_1 = [-1, 1]^T\)
Romeo-Juliet

- A surreal love affair with a twisted story
Romeo-Juliet

- A surreal love affair with a twisted story
- Verbal statement: Romeo loves Juliet when she loves him, he backs off when she doesn’t love him; Juliet is protective when Romeo loves her, but will approach him when he walks away.

Mathematical model:

\[
\begin{align*}
    dR \, dt &= aJ, \\
    dJ \, dt &= -bR,
\end{align*}
\]

where \(a, b > 0\).

Differentiate left eq. w.r.t. \(t\) and use the right eq. we get

\[
d^2R \, dt^2 + abR = 0
\]

Close analogy with the spring-mass problem

Indeed, many physically motivated problems have analogy, so we study a class of problems.
Romeo-Juliet

• A surreal love affair with a twisted story
• Verbal statement: Romeo loves Juliet when she loves him, he backs off when she doesn’t love him; Juliet is protective when Romeo loves her, but will approach him when he walks away.
• Mathematical model:

\[ \frac{dR}{dt} = aJ, \quad \frac{dJ}{dt} = -bR, \quad a, b > 0 \]
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  \]
- Close analogy with the spring-mass problem
- Indeed, many physically motivated problems have analogy, so we study a class of problems
Analytic solutions

- From physics, solutions are sine and cosine functions.

Let

\[ R = c_1 \sin(k t) + c_2 \cos(k t), \]

plugging into the differential equation,

\[ k = \sqrt{ab}, \]

are the initial conditions that prescribe the individual love with each other.

The problem has two degrees of freedom.

Physical interpretation: Romeo and Juliet will NEVER be together, unless initially together.

What's worse: unstable — or, marginally stable.

Let's handle this with complex exponentials.

Let

\[ R = c_1 e^{i \sqrt{ab} t} + c_2 e^{-i \sqrt{ab} t} \]

Same solution, but advantageous (in a moment we see this).
Analytic solutions

- From physics, solutions are sine and cosine functions
- Let $R = c_1 \sin(kt) + c_2 \cos(kt)$, plugging into the differential equation, $k = \sqrt{ab}$
Analytic solutions

- From physics, solutions are sine and cosine functions
- Let \( R = c_1 \sin(kt) + c_2 \cos(kt) \), plugging into the differential equation, \( k = \sqrt{ab} \)
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Physical interpretation: Romeo and Juliet will NEVER be together, unless initially together

What’s worse: unstable — or, marginally stable

Let’s handle this with complex exponentials

Let \( R = c e^{kt} \rightarrow k^2 + ab = 0 \rightarrow k = \pm i \sqrt{ab} \rightarrow R = c_1 e^{i \sqrt{ab} t} + c_2 e^{-i \sqrt{ab} t} \)

Same solution, but advantageous (in a moment we see this)
Analytic solutions

- From physics, solutions are sine and cosine functions.
- Let $R = c_1 \sin(kt) + c_2 \cos(kt)$, plugging into the differential equation, $k = \sqrt{ab}$.
- $c_1, c_2$ are the *initial conditions* that prescribes the individual love with each other.
- The problem has two degrees of freedom.

Physical interpretation: Romeo and Juliet will NEVER be together, unless initially together.

What's worse: unstable — or, marginally stable.

Let's handle this with complex exponentials.

$$R = c_1 e^{kt} + c_2 e^{-it}$$

$$R = c_1 e^{i\sqrt{ab}t} + c_2 e^{-i\sqrt{ab}t}$$

Same solution, but advantageous (in a moment we see this).
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Let $R = c_1 e^{ikt} + c_2 e^{-ikt} \rightarrow k^2 + ab = 0 \rightarrow k = \pm i\sqrt{ab} \rightarrow R = c_1 e^{i\sqrt{ab}t} + c_2 e^{-i\sqrt{ab}t}$

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- What’s worse: unstable — or, marginally stable
- Let’s handle this with complex exponentials
- Let \( R = ce^{kt} \)

\[
\rightarrow k^2 + ab = 0 \rightarrow k = \pm i\sqrt{ab} \rightarrow R = c_1 e^{i\sqrt{ab}t} + c_2 e^{-i\sqrt{ab}t}
\]
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Let’s try to make them happily ever after

- What indicates that R and J are TOGETHER? \((R = J = 0)\)
Let’s try to make them happily ever after

- What indicates that R and J are TOGETHER? \( R = J = 0 \)
- What is needed from the exponential functions? (negative real part)

Let's still work on the linear model but introduce

\[
\begin{align*}
\frac{dR}{dt} &= cR + aJ, \\
\frac{dJ}{dt} &= -bR + dj,
\end{align*}
\]

\( a, b > 0, c, d \) arbitrary

The second order equation:

\[
\ddot{R} - (c + d) \dot{R} + (ab + cd) R = 0
\]

Let \( R = ce^{kt} \rightarrow k^2 - (c + d)k + (ab + cd) = 0 \)

\[ k = \frac{(c + d) \pm \sqrt{(c + d)^2 - 4(ab + cd)}}{2} \]

At least need \( c + d < 0 \) to damp. Physical meaning?
Let’s try to make them happily ever after

- What indicates that R and J are TOGETHER? ($R = J = 0$)
- What is needed from the exponential functions? (negative real part)
- Let’s still work on the linear model but introduce
  \[
  \frac{dR}{dt} = cR + aJ, \quad \frac{dJ}{dt} = -bR + dJ, \quad a, b > 0, c, d \text{ arbitrary}
  \]
Let’s try to make them happily ever after

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\]

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\]

- The second order equation:

\[
\ddot{R} - (c + d)\dot{R} + (ab + cd)R = 0
\]

- Let \(R = ce^{kt} \rightarrow k^2 - (c + d)k + (ab + cd) = 0\)
  \(\rightarrow k = [(c + d) \pm \sqrt{(c + d)^2 - 4(ab + cd)}]/2\)
Let’s try to make them happily ever after

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- What is needed from the exponential functions? (negative real part)
- Let’s still work on the linear model but introduce
  \[
  \frac{dR}{dt} = cR + aJ, \quad \frac{dJ}{dt} = -bR + dJ, \quad a, b > 0, c, d \text{ arbitrary}
  \]
- The second order equation:
  \[
  \ddot{R} - (c + d)\dot{R} + (ab + cd)R = 0
  \]
  
  Let \( R = ce^{kt} \) → \( k^2 - (c + d)k + (ab + cd) = 0 \)
  → \( k = [(c + d) \pm \sqrt{(c + d)^2 - 4(ab + cd)}]/2 \)
  
  At least need \( c + d < 0 \) to damp. Physical meaning?
Same idea on differential equations

- For the love circle,

\[
\begin{pmatrix}
\dot{R} \\
\dot{J}
\end{pmatrix} = \begin{pmatrix} c & a \\ -b & d \end{pmatrix} \begin{pmatrix} R \\ J \end{pmatrix}
\]
Same idea on differential equations

- For the love circle,

\[
\begin{pmatrix}
\dot{R} \\
\dot{J}
\end{pmatrix} = \begin{pmatrix} c & a \\ -b & d \end{pmatrix} \begin{pmatrix} R \\ J \end{pmatrix}
\]

- We characterize rates and directions with eigenvalues/vectors
Same idea on differential equations

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\[
\begin{pmatrix}
\dot{R} \\
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\end{pmatrix} = \begin{pmatrix}
c & a \\
-b & d
\end{pmatrix}
\begin{pmatrix}
R \\
J
\end{pmatrix}
\]

- We characterize rates and directions with eigenvalues/vectors.

- Special directions are just those where velocity vector has the same direction as the position vector. (eg: \(\dot{x} = x, \dot{y} = -y\); \(\dot{x} = y, \dot{y} = x\))
Same idea on differential equations

- For the love circle,

\[
\begin{pmatrix}
\dot{R} \\
\dot{J}
\end{pmatrix} = \begin{pmatrix} c & a \\ -b & d \end{pmatrix} \begin{pmatrix} R \\ J \end{pmatrix}
\]

- We characterize rates and directions with eigenvalues/vectors

- Special directions are just those where velocity vector has the same direction as the position vector. (e.g.: \( \dot{x} = x \), \( \dot{y} = -y \); \( \dot{x} = y \), \( \dot{y} = x \))

- Find the eigenvalues and eigenvectors for the love circle problem
May 22\textsuperscript{nd}

Lecture 4: Nonlinear 2D problems
Rabbit Vs. Sheep

- Rabbit and sheep both eat grass in a closed lawn, so they form a competition.

\[ \dot{r} = r(3 - r - 2s) \quad \leftarrow F(r, s) \]
\[ \dot{s} = s(2 - r - s) \quad \leftarrow G(r, s) \]

Find all fixed points, analyze the dynamics and interpret the physical meanings.
Rabbit Vs. Sheep

- Rabbit and sheep both eat grass in a closed lawn, so they form a competition
- Both species have logistic dynamics in each, but there is also an interaction between the species

\[
\begin{align*}
\dot{r} &= r(3 - r - 2s) \\
\dot{s} &= s(2 - r - s)
\end{align*}
\]

Find all fixed points, analyze the dynamics and interpret the physical meanings.
Rabbit Vs. Sheep

- Rabbit and sheep both eat grass in a closed lawn, so they form a competition.
- Both species have logistic dynamics in each, but there is also an interaction between the species.
- When rabbit and sheep compete, sheep wins very often, but occasionally rabbits can still get something.

A simple model:

\[
\begin{align*}
\dot{r} &= r(3 - r - 2s) \\
\dot{s} &= s(2 - r - s)
\end{align*}
\]

Find all fixed points, analyze the dynamics and interpret the physical meanings.
Rabbit Vs. Sheep

- Rabbit and sheep both eat grass in a closed lawn, so they form a competition.
- Both species have logistic dynamics in each, but there is also an interaction between the species.
- When rabbit and sheep compete, sheep wins very often, but occasionally rabbits can still get something.
- A simple model

\[
\begin{align*}
\dot{r} &= r(3 - r - 2s) \leftarrow F(r, s) \\
\dot{s} &= s(2 - r - s) \leftarrow G(r, s)
\end{align*}
\]
Rabbit Vs. Sheep

- Rabbit and sheep both eat grass in a closed lawn, so they form a competition.
- Both species have logistic dynamics in each, but there is also an interaction between the species.
- When rabbit and sheep compete, sheep wins very often, but occasionally rabbits can still get something.
- A simple model

\[
\begin{align*}
\dot{r} &= r(3 - r - 2s) \leftarrow F(r, s) \\
\dot{s} &= s(2 - r - s) \leftarrow G(r, s)
\end{align*}
\]

- Find all fixed points, analyze the dynamics and interpret the physical meanings.
The Jacobian

- The Jacobian captures local dynamics by linearization

\[
\begin{align*}
\dot{\mathbf{r}} &= \mathbf{F}(\mathbf{r}, \mathbf{s}) + \mathbf{J}_r(\mathbf{r} - \mathbf{r}^*) + \mathbf{J}_s(\mathbf{s} - \mathbf{s}^*) \\
\dot{\mathbf{s}} &= \mathbf{G}(\mathbf{r}, \mathbf{s}) + \mathbf{J}_r(\mathbf{r} - \mathbf{r}^*) + \mathbf{J}_s(\mathbf{s} - \mathbf{s}^*)
\end{align*}
\]

\[
\left(\begin{array}{c}
\dot{\mathbf{r}} \\
\dot{\mathbf{s}}
\end{array}\right) = \left(\begin{array}{cc}
\mathbf{F}_r & \mathbf{F}_s \\
\mathbf{G}_r & \mathbf{G}_s
\end{array}\right)
\]

Local dynamics governed by the Jacobian matrix \[
\mathbf{J} = \left(\begin{array}{cc}
\mathbf{F}_r & \mathbf{F}_s \\
\mathbf{G}_r & \mathbf{G}_s
\end{array}\right)
\]
The Jacobian

- The Jacobian captures local dynamics by linearization.
- Concept: on fixed points \((r^*, s^*)\), \(\dot{r}|_{r^*} = 0, \dot{s}|_{s^*} = 0\), so locally \(\xi = r - r^*, \eta = s - s^*\) and \(\dot{r} = \dot{\xi} - r^* = \dot{\xi}, \dot{s} = \dot{\eta} - s^* = \dot{\eta}\).
The Jacobian

- The Jacobian captures local dynamics by linearization.

- Concept: on fixed points \((r^*, s^*)\), \(\dot{r} \mid_{r^*} = 0, \dot{s} \mid_{s^*} = 0\), so locally
  \[\xi = r - r^*, \eta = s - s^*\] and \[\dot{r} = \dot{\xi} - r^*, \dot{s} = \dot{\eta} - s^* = \dot{\eta}\]

- On RHS, linearize \(F, G\) around \((r^*, s^*)\)

\[
F \approx F(r^*, s^*) + F_r(r - r^*) + F_s(s - s^*) = 0 + F_r(r - r^*) + F_s(s - s^*)
\]
\[
G \approx G(r^*, s^*) + G_r(r - r^*) + G_s(s - s^*) = 0 + G_r(r - r^*) + G_s(s - s^*)
\]

\[
\begin{pmatrix}
\dot{\xi} \\
\dot{\eta}
\end{pmatrix} = 
\begin{pmatrix}
F_r & F_s \\
G_r & G_s
\end{pmatrix}
\begin{pmatrix}
\xi \\
\eta
\end{pmatrix}
\]
The Jacobian captures local dynamics by linearization.

Concept: on fixed points \((r^*, s^*)\), \(\dot{r}|_{r^*} = 0, \dot{s}|_{s^*} = 0\), so locally
\[
\xi = r - r^*, \eta = s - s^* \quad \text{and} \quad \dot{r} = \dot{\xi} - r^* = \dot{\xi}, \dot{s} = \dot{\eta} - s^* = \dot{\eta}
\]

On RHS, linearize \(F, G\) around \((r^*, s^*)\)

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F \approx F(r^*, s^*) + F_r(r - r^*) + F_s(s - s^*) = 0 + F_r(r - r^*) + F_s(s - s^*)
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G \approx G(r^*, s^*) + G_r(r - r^*) + G_s(s - s^*) = 0 + G_r(r - r^*) + G_s(s - s^*)
\]

\[
\begin{pmatrix}
\ddot{\xi} \\
\ddot{\eta}
\end{pmatrix} =
\begin{pmatrix}
F_r & F_s \\
G_r & G_s
\end{pmatrix}
\begin{pmatrix}
\xi \\
\eta
\end{pmatrix}
\]

Local dynamics governed by the Jacobian matrix

\[
\begin{pmatrix}
F_r & F_s \\
G_r & G_s
\end{pmatrix}
\]
Fixed points and linearization

- From governing equations,

\[ r = 0 \text{ or } r = 3 - 2s \]
\[ s = 0 \text{ or } r = 2 - s \]
Fixed points and linearization

- From governing equations,
  
  \[ r = 0 \quad \text{or} \quad r = 3 - 2s \]
  \[ s = 0 \quad \text{or} \quad r = 2 - s \]

- Let \( \mathbf{x} = (r, s) \), 4 fixed points: \((0, 0), (0, 2), (3, 0), (1, 1)\) where \( \dot{x}^* = 0 \)
Fixed points and linearization

- From governing equations,
  
  \[ r = 0 \quad \text{or} \quad r = 3 - 2s \]
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- Local dynamics captured by the Jacobian

\[
\begin{pmatrix}
3 - 2r - 2s & -2r \\
-s & 2 - r - 2s
\end{pmatrix}
\begin{pmatrix}
\xi \\
\eta
\end{pmatrix}
\]
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\xi \\
\eta
\end{pmatrix}
\]

- Problem becomes eigenvalues/eigenvectors of the Jacobian matrix
Stability of fixed points

- At (0, 0), \( A = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}, \ k_1 = 3, \ v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \ k_2 = 2, \ v_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \)
Stability of fixed points

- At (0, 0), $A = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$, $k_1 = 3$, $v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $k_2 = 2$, $v_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

- At (3, 0), $A = \begin{bmatrix} -3 & -6 \\ 0 & -1 \end{bmatrix}$, $k_1 = -3$, $v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $k_2 = -1$, $v_2 = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$
Stability of fixed points

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Stability of fixed points

- At \((0, 0)\), \(A = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}\), \(k_1 = 3\), \(v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}\), \(k_2 = 2\), \(v_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}\)

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- At \((1, 1)\), \(A = \begin{bmatrix} -1 & -2 \\ -1 & -1 \end{bmatrix}\), \(k_1 = \sqrt{2} - 1\), \(v_1 = \begin{bmatrix} \sqrt{2} \\ -1 \end{bmatrix}\), \(k_2 = -\sqrt{2} - 1\), \(v_2 = \begin{bmatrix} \sqrt{2} \\ 1 \end{bmatrix}\)
Stability of fixed points

- At (0, 0), \( A = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix} \), \( k_1 = 3 \), \( v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \), \( k_2 = 2 \), \( v_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \)

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- Phase portrait
Limit cycles in love affair

- Let's look at a more complicated love cycle problem

\[ \dot{R} = J + aR(R^2 + J^2) \]
\[ \dot{J} = -R + aJ(R^2 + J^2) \]
Limit cycles in love affair

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- Linearization still shows a marginally stable cycle
- Contribution of the cubic terms is going to determine the fate of their love

Let’s convert to polar coordinates!

\[ R = r \cos \theta, \quad J = r \sin \theta \]

So \[ \dot{r} = ar^3, \quad \dot{\theta} = -1 \]

We can even make their love circle structurally stable!

Try \[ \dot{r} = ar(1 - r^2), \quad \dot{\theta} = 1 \], unstable at \( r = 0 \), stable at \( r = 1 \). Can you write this under Cartesian coordinate?
Limit cycles in love affair

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- \(R^2 + J^2 \to \text{polar coordinates!}\)
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- Contribution of the cubic terms is going to determine the fate of their love
- \( R^2 + J^2 \to \) polar coordinates!
- \( R = r \cos \theta, J = r \sin \theta \) so \( \dot{r} = ar^3, \dot{\theta} = -1 \)

![Graphs showing different behaviors for different values of a](image-url)
Limit cycles in love affair

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Nonlinear pendulum

- Let's look at the dynamics of a REAL pendulum, not the small angle approximation.
Nonlinear pendulum

- Let’s look at the dynamics of a REAL pendulum, not the small angle approximation
- Discuss the formulation of problem by analyzing forces

\[ \ddot{\theta} + \frac{g}{L} \sin(\theta) = 0 \]

Analyze the fixed points \((0, 0), (\pi, 0)\), their stability (marginally stable, unstable)

Phase portrait, direction of trajectories
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The resulting equation: $\ddot{\theta} + g/L \sin(\theta) = 0$
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Damped nonlinear pendulum

- Adding some damping corresponding to friction forces

The governing equation:

\[ \ddot{\theta} + \frac{\gamma}{m} \dot{\theta} + \frac{g}{L} \sin(\theta) = 0 \]

Analyze the fixed points by linearization (0, 0), (\pi, 0), their stability (stable, unstable)
Damped nonlinear pendulum

- Adding some damping corresponding to friction forces
- The governing equation:

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- Analyze the fixed points by linearization (0, 0), (\(\pi\), 0), their stability (stable, unstable)
- Phase portrait
May 23rd
Lecture 5: Sea ice dynamics, a 2D version
Formulation

- Assume a simple sea ice recession-progression model based on two parameters, $L$ and $C$
Formulation

- Assume a simple sea ice recession-progression model based on two parameters, $L$ and $C$
- The hypothetical cycle:
  Progression (L decrease) $\rightarrow$ Less ocean surface $\rightarrow$
  More carbon in air $\rightarrow$ Temperature increase $\rightarrow$
  Melt ice, recession $\rightarrow$ More ocean surface $\rightarrow$
  Less carbon in air $\rightarrow$ Temperature decrease $\rightarrow$ Progression
Formulation

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  - Progression (L decrease) → Less ocean surface → More carbon in air → Temperature increase → Melt ice, recession → More ocean surface → Less carbon in air → Temperature decrease → Progression
- Seems like a simple spring-mass oscillation but — on sphere!
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  - Melt ice, recession $\rightarrow$ More ocean surface $\rightarrow$
  - Less carbon in air $\rightarrow$ Temperature decrease $\rightarrow$ Progression
- Seems like a simple spring-mass oscillation but — on sphere!
- Coordinate:
Formulation

- Assume North-South symmetry, azimuthal invariance and perfect mixture of carbon
Formulation

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- Excessive amount of carbon is linearly proportional to the volume of ice melt — sea surface area change
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- Assume an equilibrium state where Latitude of ice cap and carbon concentration remain constant
Formulation

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- Excessive amount of carbon is linearly proportional to the volume of ice melt — sea surface area change
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- Assume an equilibrium state where Latitude of ice cap and carbon concentration remain constant
- The simplest Cartesian coordinate analogy:

\[
\frac{dL}{dt} = a(C - C_0) \\
\frac{dC}{dt} = -b(L - L_0)
\]
Formulation

- Assume North-South symmetry, azimuthal invariance and perfect mixture of carbon
- Excessive amount of carbon is linearly proportional to the volume of ice melt — sea surface area change
- Sea surface area away from equilibrium is linearly proportional to reduction of carbon
- Assume an equilibrium state where Latitude of ice cap and carbon concentration remain constant
- The simplest Cartesian coordinate analogy:
  \[
  \frac{dL}{dt} = a(C - C_0)
  \]
  \[
  \frac{dC}{dt} = -b(L - L_0)
  \]
- Our model is a deviation from this simple formulation
The Latitude equation

- Volume change of ice strip:
  \[ RdL \times 2\pi r \times thickness(100m) = 200\pi R^2 \cos LdL \]
The Latitude equation

- Volume change of ice strip:
  \[ RdL \times 2\pi r \times \text{thickness}(100\,m) = 200\pi R^2 \cos LdL \]

- Change of carbon concentration over time (change of carbon substance in air): \( a(C - C_0)dt \)
The Latitude equation

- Volume change of ice strip:
  \[ RdL \times 2\pi r \times \text{thickness}(100m) = 200\pi R^2 \cos L dL \]

- Change of carbon concentration over time (change of carbon substance in air): \( a(C - C_0)dt \)

Thus:

\[
\frac{dL}{dt} = \frac{a'(C - C_0)}{\cos L} \rightarrow F
\]
The Carbon equation

- Sea surface area surplus/deficit:

\[ A = \int_{L_0}^{L} 2\pi r R dL' = \int_{L_0}^{L} 2\pi R^2 \cos L' dL' = 2\pi R^2 (\sin L - \sin L_0) \]
The Carbon equation

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- Thus

\[ \frac{dC}{dt} = -bA = -b'(\sin L - \sin L_0) \rightarrow G \]
Dynamics near fixed point

- Do we still expect a circle near \((L_0, C_0)\)?
Dynamics near fixed point

- Do we still expect a circle near \((L_0, C_0)\)?
- Jacobian:

\[
A = \begin{bmatrix}
a'(C - C_0)[-\sin L/\cos L^2] & a'/\cos L \\
-b'/\cos L & 0
\end{bmatrix} \bigg|_{L_0, C_0}
\]

\[
= \begin{bmatrix}
0 & a'/\cos L_0 \\
-b'/\cos L_0 & 0
\end{bmatrix}
\]
Dynamics near fixed point

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\end{bmatrix}
\mid_{L_0, C_0}
\]

\[
= \begin{bmatrix}
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\end{bmatrix}
\]

- Linearization doesn’t tell, need higher order terms — maybe we need polar coordinate!
Dynamics near fixed point

- Do we still expect a circle near \((L_0, C_0)\)?

- Jacobian:

\[
A = \left[ \begin{array}{cc}
a'(C - C_0)[ - \sin L/ \cos L^2 ] & a'/ \cos L \\
- b' \cos L & 0
\end{array} \right] \bigg|_{L_0, C_0}
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\[
= \left[ \begin{array}{cc}
0 & a'/ \cos L_0 \\
- b' \cos L_0 & 0
\end{array} \right]
\]

- Linearization doesn’t tell, need higher order terms — maybe we need polar coordinate!

- Let’s try a little in this direction by expanding \(F, G\) into higher order terms
Numerical predictions

- Use numerical simulations to see what happens, choose
  \( L_0 = \pi/3, C_0 = 1, a' = 0.1, b' = -0.1 \)

![Graph showing numerical predictions](image)
Numerical predictions

- Use numerical simulations to see what happens, choose $L_0 = \pi/3$, $C_0 = 1$, $a' = 0.1$, $b' = -0.1$
Numerical predictions

- Use numerical simulations to see what happens, choose $L_0 = \pi/3$, $C_0 = 1$, $a' = 0.1$, $b' = -0.1$

- Using these parameters to check into our prediction on the dynamics?
Additional constraints and physical interpretations

- Physically we don’t expect $C$ to drop below 0, $L$ is between equator and the pole.
Additional constraints and physical interpretations

- Physically we don’t expect $C$ to drop below 0, $L$ is between equator and the pole
- Long term dynamics on phase plane
Additional constraints and physical interpretations

- Physically we don’t expect $C$ to drop below 0, $L$ is between equator and the pole.
- Long term dynamics on phase plane.

Physical meaning of the data: ice-free — snowball in short time.
May 23rd

Lecture 6: Sea-ice dynamics, 3D version
Formulation

- The carbon-sea ice interaction is rather indirect
Formulation

- The carbon-sea ice interaction is rather indirect
- Factor in temperature interactions with the rest

\[ \frac{dC}{dt} = -bA = -b' (\sin L - \sin L_0) \]

Melting (change in latitude) directly linked to temperature

\[ \frac{dL}{dt} = a' (T - T_0) \cos L \]

Contributions to temperature change: solar input, reflection of heat from ice/ocean surface
Formulation

- The carbon-sea ice interaction is rather indirect
- Factor in temperature interactions with the rest
- Surface absorption of carbon — unchanged

\[
\frac{dC}{dt} = -bA = -b'(\sin L - \sin L_0)
\]
Formulation

- The carbon-sea ice interaction is rather indirect
- Factor in temperature interactions with the rest
- Surface absorption of carbon — unchanged
  \[ \frac{dC}{dt} = -bA = -b'(\sin L - \sin L_0) \]
- Melting (change in latitude) directly link to temperature
  \[ \frac{dL}{dt} = a'(T - T_0) \cos L \]
The carbon-sea ice interaction is rather indirect
Factor in temperature interactions with the rest
Surface absorption of carbon — unchanged
\[
\frac{dC}{dt} = -bA = -b'(\sin L - \sin L_0)
\]
Melting (change in latitude) directly link to temperature
\[
\frac{dL}{dt} = a'(T - T_0) / \cos L
\]
Contributions to temperature change: solar input - reflection of heat from ice/ocean surface
The temperature equation

- Assume constant heat flux straight into earth equator $q$
The temperature equation

- Assume constant heat flux straight into earth equator \( q \)
- Effective heating normal to earth surface at each latitude \( q \cos L \)
The temperature equation

- Assume constant heat flux straight into earth equator \( q \)
- Effective heating normal to earth surface at each latitude \( q \cos L \)

Integrated heating

\[
\int_{0}^{\pi/2} q \cos L' 2\pi R \cos L' RdL'
\]
The temperature equation

- Assume constant heat flux straight into earth equator \( q \)
- Effective heating normal to earth surface at each latitude \( q \cos L \)

- Integrated heating \( \int_{0}^{\pi/2} q \cos L' 2\pi R \cos L' RdL' \)
- Reflection due to albedo
  \(- \int_{0}^{L} a_o \cos L' 2\pi R \cos L' RdL' - \int_{L}^{\pi/2} a_i \cos L' 2\pi R \cos L' RdL' \)
The temperature equation

- Assume constant heat flux straight into earth equator $q$
- Effective heating normal to earth surface at each latitude $q \cos L$

- Integrated heating $\int_0^{\pi/2} q \cos L' 2\pi R \cos L' RdL'$
- Reflection due to albedo $- \int_0^L a_o \cos L' 2\pi R \cos L' RdL' - \int_L^{\pi/2} a_i \cos L' 2\pi R \cos L' RdL'$
- Heat increase due to excess carbon $\eta (C - C_0)$, hence

$$\frac{dT}{dt} = \eta(C - C_0) + 2\pi R^2 q \left( \int_0^{\pi/2} \cos^2 L' dL' - a_o \int_0^L \cos^2 L' dL' - a_i \int_L^{\pi/2} \cos^2 L' dL' \right)$$

$$= \eta(C - C_0) + 2\pi R^2 q (1 - a_o) \frac{\pi}{4} - 2\pi R^2 q (a_i - a_o) \left( \frac{\pi}{4} - \frac{L}{2} - \frac{\sin 2L}{4} \right)$$
Numerical predictions

- Use numerical simulations to see what happens, choose
  \[ T_0 = 1, \eta = 0.1, a_i = a_o, 2\pi R^2 q(1 - a_o)\frac{\pi}{4} = 0.005\pi/4 \]
Numerical predictions

- Use numerical simulations to see what happens, choose $T_0 = 1, \eta = 0.1, a_i = a_o, 2\pi R^2 q(1 - a_o) \frac{\pi}{4} = 0.005\pi/4$

- What can we infer on the dynamics?