Lagrangian coherent structures and internal wave attractors

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For a nonuniformly stratified layer of fluid, internal gravity waves propagate at varying angles depending on the local buoyancy and Coriolis (in geophysical applications) frequencies. Relatively confined geometries, such as multiple submarine ridges, can support internal wave attractors, which can be viewed as Lagrangian coherent structures for the energy density flux. Since traditional approaches for locating these structures prove cumbersome, here we develop an approach that efficiently extracts the locations of internal wave attractors, as well as quantifying the rate of attraction. Using realistic geometry and stratification from ocean observations, we find that a significant northern portion of the Luzon ridge can support internal wave attractors. © 2010 American Institute of Physics. [doi:10.1063/1.3273054]

I. INTRODUCTION

Internal gravity wave energy propagates along wave characteristics and is widely believed to be responsible for intensive mixing in the ocean when internal waves break. Recently, studies have revealed the possibility of internal wave attractors in confined basins and between multiple submarine ridges that have steep slopes, providing a new mechanism for strong mixing to occur away from ocean boundaries. These recent studies have focused on linear stratifications in ideal laboratory conditions. Motivated by recent ideas concerning Lagrangian coherent structures (LCS), in this paper we present an efficient mathematical tool that uses Lyapunov exponents to extract internal wave attractors in nonuniformly stratified fluids. This approach remedies the inherent difficulty of constructing a return map where there is complex topography and skewing of internal wave characteristics due to a changing buoyancy frequency.

Internal gravity waves are ubiquitous in density stratified fluids and their generation can be attributed to many physical processes. In the ocean, a primary source of internal waves is the periodic sloshing of the barotropic (depth-independent) tide over rough bottom topography. The linear response of the ocean to this forcing is the generation of baroclinic (depth-dependent) modes known as internal tides, which exhibit periodic motion at the tidal frequency. Recent observational and laboratory experimental studies indicate that the propagation of internal tides carries energy away from generation sites and may lead to intensive mixing in the ocean interior.\textsuperscript{2,5,7,16,18} Being able to locate and quantify the internal wave energy is thus imperative in modeling the interaction between internal tides and other ocean processes.

Internal wave energy propagates in a peculiar fashion as compared with Snell’s law for optic rays. Due to the dispersion relation,\textsuperscript{6} the angle of internal wave characteristics with respect to the horizontal $\theta$ is dictated by the intrinsic tidal frequency $\omega$ and the local buoyancy and Coriolis frequencies $N$ and $f$, respectively. This dispersion relation has several implications on the propagation of internal wave energy. First, since internal wave characteristics preserve their angle with respect to the vertical axis, there is focusing and defocusing of wave phases when internal waves meet a slanted bottom slope; internal wave energy therefore converges or diverges upon each of such reflections. Second, internal wave characteristics can reflect with respect to either the vertical or the horizontal directions, depending on the slope of a wave characteristic and the slope of the topography from which reflection is taking place. Third, in the ocean interior, where fluid density is nonuniformly stratified, changing buoyancy frequency will bend otherwise straight characteristics, leading to a skewed channel of concentrated wave energy.

For internal waves to reflect horizontally, the topographic slope must be shallower than the inclination of the internal wave characteristic; such a topography is referred to as “subcritical.” Conversely, “supercritical” topography has slopes steeper than the wave characteristic inclination, so waves reflect vertically. In a closed, two-dimensional (2D) ocean basin with only subcritical bottom topography, the corners of the basin act as attractors to internal wave characteristics.\textsuperscript{19,20} In this setting, internal wave characteristics propagating toward either corner get continuously focused by multiple reflections and meet their destiny when they break at the corners. A significant portion of the ocean floor is not subcritical, however, which allows internal waves to be reflected back into the ocean interior and possibly form geometric structures referred to as “internal wave attractors,”\textsuperscript{12} which continuously focus internal wave energy onto specific closed ray paths. Laboratory experiments have demonstrated that attractors do exist in enclosed trapezoidal basins.\textsuperscript{9,11} Recently, Echeverri and Peacock\textsuperscript{3} and Echeverri \textit{et al.}\textsuperscript{4} reported the existence of such attractors in systems of...
II. MATHEMATICAL BACKGROUND

The equation governing the motion of a 2D linear, inviscid, nonhydrostatic, stratified, Boussinesq fluid in a rotating frame is

\[(N^2 + \partial_x \psi_{xx} + (f^2 + \partial_z \partial_z) \psi_{zz} = 0, \tag{1}\]

where \(\psi\) is the stream function, \(x\) and \(z\) are the horizontal and vertical coordinates, \(t\) is time, \(f\) is the constant Coriolis parameter, subscripts denote the partial derivatives, and \(N^2 = -(g/\rho_0) d\rho(z)/dz\) is the buoyancy frequency, with \(g\) the acceleration due to gravity, \(\rho(z)\) the background density, and \(\rho_0\) a reference density.

For a monochromatic wave of frequency \(\omega\), \(\psi\) takes the form of

\[\psi = \Re[e^{-i\omega t} \hat{\psi}(x, z)], \tag{2}\]

and Eq. (1) becomes

\[(N^2 - \omega^2) \psi_{xx} = (\omega^2 - f^2) \psi_{zz}. \tag{3}\]

An important requirement for a traveling wave solution to exist is that the tidal frequency \(\omega\) must be between the buoyancy and Coriolis frequencies. As an example, the semidiurnal lunar tide (M2) has a frequency of about 1/12.4 cycle/h, the typical buoyancy frequency in the ocean is 1 cycle/h, and the Coriolis frequency \(f\) varies with latitude \(\phi\) according to \(f = \Omega \sin \phi\), where \(\Omega\) is the frequency of earth rotation (1/24 cycle/h). As such, semidiurnal tides only propagate below 75° latitude.

When the density stratification is uniform, the buoyancy frequency \(N^2\) is constant. In such cases, internal wave characteristics have a constant inclination. Assuming that the ocean surface \(z=0\) is a rigid lid, the ocean floor has a constant depth \(-h_0\) away from the topography of interest, whose depth is \(h(x)\), and the characteristic velocity scale of the tidal flow is \(U\), one can nondimensionalize the problem by

\[Z = \frac{z}{h_0}, \quad X = \frac{x}{\mu h_0}, \quad H(X) = \frac{h(\mu h_0 X)}{h_0}, \quad \Psi = \frac{\hat{\psi}}{Uh_0}, \tag{4}\]

where \(\mu = \sqrt{(N^2 - \omega^2)/(\omega^2 - f^2)}\) is the inverse ray slope. In the ocean, however, the density stratification is nonuniform, and thus the buoyancy frequency \(N^2\) is a function of \(z\). In order to trace out the characteristics using ray tracing in the physical coordinate, one would have to compute the wave angle at each depth because of the varying stratification. Alternatively, one can work with a vertically stretched coordinate usually used in Wentzel–Kramers–Brillouin (WKB) approximation (e.g., Ref. 10)

\[\tilde{N} = \int_{-h_0}^{0} \frac{N(z')}{h_0} dz', \quad Z = \int_{-h_0}^{z} \frac{N(z')}{\tilde{N}h_0} dz' - 1, \quad H = \int_{-h_0}^{h} \frac{N(z')}{\tilde{N}h_0} dz' - 1, \quad \mu = \sqrt{\frac{N^2 - \omega^2}{\omega^2 - f^2}}, \quad X = \frac{x}{\mu h_0}. \tag{5}\]

The nondimensional vertical coordinate \(Z\) is chosen such that the ocean bottom is \(-1\) and ocean surface is \(0\). Note that we do not explicitly determine the stream function \(\Psi\), and thus
WKB approximation is not really invoked. In both cases, the
nondimensional governing equation reduces to
\[ \Psi_{XX} - \Psi_{ZZ} = 0, \quad (6) \]
where \( X \) and \( Z \) are the appropriate nondimensional stretched coordinates and the ocean has unit depth. The factor \( \pi \) reduction in ocean depth compared with Llewellyn Smith and Young\(^1\) and Echeverri et al.\(^4\) is chosen for convenience. In these redefined coordinate systems, internal wave characteristics propagate at 45°. Simple internal wave attractors, therefore, take the form of rectangular orbits between double ridge systems (cf. Fig. 2).

Note that we have been treating the propagation of internal wave energy density as material particles and their dynamics are governed by the Lagrangian motion of the characteristic trajectories. In this respect, the limit cycles we extract are steady, infinite-time invariant manifolds that dictate the organization of energy pathways in the ocean interior, and thus they are LCS. However, direct extraction using a standard LCS extraction technique, such as the finite-time Lyapunov exponents,\(^5,17\) was found to be inefficient.

In the steady state, an internal wave attractor serves as the \( \Omega \)-limit set (stable limit cycle) that attract nearby trajectories—neighboring characteristics approach this limit cycle asymptotically in a given direction. Since two characteristics will propagate parallel to each other in the fluid interior, and convergence/divergence only takes place when the characteristics reflect at the ocean floor when inclined, it is convenient to just examine the propagation of initial conditions from the ocean surface. We use a map that iterates an initial condition at the ocean surface to its next surface location. By analyzing the dynamics of intersections of trajectories with the ocean surface, we obtain the dynamics of limit cycles—the ocean surface is the Poincaré section and the intersections between the attractor and the surface are stable fixed points which attract nearby points in the Poincaré section. To consolidate our discussion, we focus only on the subset of attractors that exist between double ridges, reflecting once on each of the interior supercritical flanks of the double ridge system. More complex systems can be studied using the methods presented here, but the double-ridge configuration is the most readily accessible and perhaps the most likely to be found in nature. Following Maas and Lam,\(^12\) we refer to this subset as \((n,1)\)-attractors, where \( n \) is the number of reflections at the ocean surface and 1 refers to a single reflection on a supercritical segment of a ridge (i.e., one reflection on the way down and one reflection on the way back up).

For a generic double ridge system, we can define the distance \( L \) between the two ridges as the nondimensional distance between the two peaks of the ridge. The number of surface reflections required for an attractor can be readily determined. In the rescaled coordinates, since internal wave inclinations are always at 45°, characteristics have to travel the same horizontal and vertical distances to form a \((1,1)\)-cycle. Therefore, ridges have to be at least one unit length apart but less than two unit lengths wide to support a \((1,1)\) attractor (i.e., \( 1 < L < 2 \)). Likewise, they have to be separated between \( n \) and \( n+1 \) unit lengths to support a \((n,1)\)-attractor (i.e., \( n < L < n+1 \)). The total number of reflections to form a closed loop for \((n,1)\)-attractors is, therefore, \( 2(n+1) \).

Once the number of reflections has been determined, we seek initial conditions at the ocean surface that attracts nearby trajectories. To locate these, we numerically map an initial condition to the ocean floor and record the next horizontal location at which it completes 2\((n+1)\) reflections and returns to the surface. This mapping is the Poincaré map of the system. The mapping between the surface and the floor is created inversely, starting from the ocean floor, with characteristics of slopes \( \pm 1 \). Take the \(-1\) characteristics as an example (cf. Fig. 1). The leftmost portion of these characteristics is mapped between the subcritical and supercritical ocean floors near the left ridge, and to the right characteristics are mapped to the ocean surface. Therefore, we locate the two critical points having a slope of \(-1\) on the left ridge. The portion of the ridge between these two points is the supercritical portion of the left ridge. The top critical point \( c_1 \) is mapped to a distant point \( c_3 \) on the ocean floor and the bottom critical point \( c_1 \) is mapped to itself. To the right of \( c_3 \), a point \((X,H)\) on the ocean floor is mapped to \((X+H,0)\) at the surface. To the left of \( c_3 \), a point \((X,H)\) and its map \((X',H')\) satisfy \( X+H = X' + H' \), and only one unique point between \( c_1 \) and \( c_3 \) can satisfy this relation (otherwise, a different point on the ocean floor is blocking the characteristics). Thus we compute \( X+H \) for every point between \( c_1 \) and \( c_3 \) and find the mapping. Following the same procedure, we also create the mapping for \(+1\) characteristics. This approach greatly reduces the computational time as compared with ray tracing, where small distances are advanced upon each numerical iteration.
After the creation of the map, we start the initial conditions from the ocean surface (either to the left or to the right will be sufficient, as we show later). By iterating only $2(n+1)$ times, for an initial condition $(X_0, 0)$ near the attractor, trajectories return to the ocean surface the $n$th time at $(X_n, 0)$. Thus we create a distance function $d(X; \alpha, \beta, \ldots)$, defined as $d(X) = X_0 - X_n$ where $\alpha, \beta, \ldots$ are parameters that control the shape of the sea-floor topography (examples of these parameters include the ridge distance $L$ and the ridge heights $H_L, H_R$ defined as the vertical distance between the ridge peaks and the deepest point in the ocean floor between the ridges).

Initial conditions having $d(X_0) = 0$ are crucial as they return to themselves after $2(n+1)$ reflections. Therefore, the trajectory associated with each such initial condition is a closed orbit. Not all closed orbits are limit cycles, however, which is what we seek to determine for an internal wave attractor. One counterexample is wave propagation in a square basin. Here, every initial condition falls on a closed orbit, but these orbits are densely populated in the basin, and thus no limit cycle exists. Furthermore, even if a closed orbit is a limit cycle, its stability type may be stable or unstable. Only stable limit cycles are wave attractors for the direction in which energy propagation is being considered. For unstable limit cycles, wave energy propagates away from the limit cycle, approaching it only in backward time or if the direction of energy propagation is reversed. Thus, for a stable limit cycle, internal wave characteristics propagating in the direction for which stability has been determined will end up on the attractor and those propagating in the opposite direction will have wave energy escaping the domain of interest. Since internal tides generated at the supercritical slopes of the double ridges can propagate in both clockwise and counterclockwise directions, both stable and unstable limit cycles are important in locating wave attractors, since a limit cycle that is unstable to clockwise traveling characteristics is stable to counterclockwise traveling characteristics. Indeed, this means that one needs only to initiate wave characteristics from initial conditions at the ocean surface in one direction to find all the attractors.

The following theorem formally defines limit cycles in the double ridge system and determines their stability.

**Theorem 1:** If the distance function $d$ has distinct zeros between the double ridges, then locations of these distinct zeros correspond to surface intersection of limit cycles, which are fixed points on the Poincaré section. If, in addition, the distance function crosses zero transversely, then the stability of the limit cycles can be determined by the slopes of $d$ at the zero locations. Specifically, if the slope is less than zero, the limit cycle is stable; if it is greater than zero, the limit cycle is unstable. On the other hand, if the slope is equal to zero (the distance function being tangent to $d = 0$), the limit cycle is semistable, attracting trajectories on one side of the limit cycle and repelling trajectories on the other side.

**Proof:** Since $d$ has distinct zeros, for each fixed point $X^*$ we can find a positive $\varepsilon$ where $d(X) = 0, X \in [X^* - \varepsilon, X^*] \cup [X^*, X^* + \varepsilon]$. As trajectories are not closed orbits ($d \neq 0$) except for the one that contains $X^*$, this trajectory is the unique closed orbit between $[X^* - \varepsilon, X^* + \varepsilon]$ and thus a limit cycle.

If $X^*$ is a transversal zero, it means that on one side of $X^*$, $d$ takes one sign, and on the other side of $X^*$, $d$ takes a different sign. For any initial condition $X_0$, the Poincaré map $P(X_0) = X_0 + d(X_0)$ maps $X_0$ to its next surface intersection after $2(n+1)$ reflections. For an initial condition $X_0$ close to the fixed point $X^*$,

$$P(X_0) = (X_0 - X^*) + d(X_0) = (X_0 - X^*) + \frac{1}{2}(X_0 - X^*)^2 d''(X^*)$$

which is tangent to $X^*$ at $X^*$, hence $X^*$ is a saddle node bifurcation, captured also by the distance function $d$.

Where $q = dq/dX$ denotes the ordinary derivative of a quantity $q$ with respect to $X$. First note that if $X_0 = X^*$, the Poincaré map of $X_0$ just returns to itself, irrespective of the signs of $d'(X^*)$, $d''(X^*)$, etc. For initial conditions away from $X^*$, if $d'(X^*) < 0, P(X_0) > X_0$ when $X_0 < X^*$ and $P(X_0) < X_0$ when $X_0 > X^*$, thus $X^*$ attracts nearby initial conditions. Similarly, we can show that if $d'(X^*) > 0$ then $X^*$ repels nearby initial conditions.

The approach described above is similar to the 1D bimodal map discussed by Maas and Lam and Echeverri et al., in that characteristics are necessarily iterated forward. However, aside from being capable of handling complex stratification and geometry, the number of iterations is greatly reduced here due to the construction of $d$ as a function of $X$. In fact, the crucial difference is that by the construction of $d$, we arrive at a rigorous definition of internal wave attractor, as compared with the hope that a large number of iterations will lead the characteristics toward the attractor. Indeed, increasing the number of iterations and observing trajectories approaching some limit point may not at all guarantee a limit cycle—if $d$ is approaching but not yet crossing zero, it may take forever for a trajectory to go through the region where $d$ is close to zero, often referred to as a “ghost.” The distance function $d$ clearly tells us precisely if a fixed point can be located.

As seen in Maas and Lam and Echeverri et al., when the topography is varied beyond some critical shape parameter value, the wave characteristics immediately fail to converge to an attractor. This change in dynamics corresponds to a saddle node bifurcation, captured also by the distance function $d$. Before $d$ crosses zero, no fixed point exists on the Poincaré section, and thus no limit cycle can be located. As the distance function touches the zero axis at one point, the system undergoes the bifurcation and the fixed point is semi-stable. By varying the parameters further such that $d$ crosses zero transversally, we observe distinct limit cycles inside the double ocean ridge system.

Since the convergence/divergence rate of nearby characteristics on the stretched bottom topography is purely determined by the topographic slope, we compute the relevant Lyapunov exponent of the attractor to characterize the rate of convergence. The Lyapunov exponent for each
FIG. 2. Two modes of wave attractors supported by double Gaussian ridges. The nondimensional parameters for the topographies are $H_L=0.9$, $H_R=0.6$, $\sigma_L=0.315$, $\sigma_R=0.2$. (a) (1,1) attractor supported by two nearby Gaussian ridges with $L=1.43$. The distance function $d(X_0, L)$ for this configuration is shown as the dotted line. For characteristics propagating to the left, a stable fixed point is located at $X=0.5299$ and an unstable fixed point is located at $X=0.6926$. They are denoted by the closed and open circles, respectively. The stable fixed point corresponds to an attractor and the unstable fixed point corresponds to a repellor. Stability type reverses when characteristics are initiated to the right. (b) (2,1) attractor supported by the double Gaussian system with $L=2.45$. Note that the stable and unstable fixed points are on the same attractor. For (2,1) attractors, we do not expect two distinct loops to emerge, as seen in (1,1) attractors illustrated in (a).

III. ATTRACTORS IN IDEAL CONFIGURATIONS

To begin, we consider the idealized case of a double Gaussian ridge. Five nondimensional parameters $L$, $H_L$, $H_R$, $\sigma_L$, and $\sigma_R$ control this family of topographies, where $L$ is the distance between the top of the ridges, $H_L$ and $H_R$ are the heights of the left and right ridges, and $\sigma_L$ and $\sigma_R$ are the variances of the Gaussian topographies on the left and right, respectively. We explore the existence and the variation in convergence rate of attractors with respect to two parameters: the distance $L$ between the two ridges and the width of one of the ridges. The other parameters are $H_L=0.9$ and $H_R=0.6$, and $\sigma_L$ is chosen such that the maximum slope angle on the left ridge is $\pi/3$. The ridge distance $L$ is examined between 1 and 3. For $L<1$, no attractor exists, and for $L>4$ it is straightforward to determine that the basins in parameter space that support attraction and the rate of convergence follow a predictable, repetitive pattern due to extra reflections on the flat ocean floor and ocean surface. More general trends of these basins upon variation of other parameters can also be inferred from these results.

For $\sigma_R=0.2$ and $L=1.43$, a (1,1) attractor is supported by the double ridge system, as shown in Fig. 2(a). For this set of parameters, we generate the distance function $d(X)$ by varying the initial conditions at the ocean surface and find that two zeros exist. Note that although initial conditions are tested throughout all points between the two ridge summits, a significant portion of them leaves the domain of interest. Only those initial conditions that return within the double ridge comprise the domain of $d$. The two zeros correspond to the stable and unstable fixed points in the Poincaré section, with the stable fixed point corresponding to an attractor and the unstable fixed point to a repellor. The two fixed points correspond to two distinct limit cycles and their stability type reverses when characteristics are initiated in the reversed direction. In Fig. 2(b) we show a (2,1) attractor when $L$ is changed to 2.45. The distance function $d$ is also shown with two zeros. For left-initiated characteristics, the fixed point on

internal wave attractor is defined as follows. Suppose two parallel characteristics are incident on a slope of angle $\kappa$ with respect to the horizontal. If the sign of the ray slope is opposite to the sign of the topographic slope, the characteristics will converge with a convergence rate $\gamma = |\tan(\kappa - \pi/4)|$, where the convergence rate denotes the rate of change in the distance between two wave characteristics upon one reflection at the ocean floor (therefore, the distance of two characteristics before the reflection is the distance of these two characteristics after the reflection divided by $\gamma$); otherwise the characteristics will diverge with a rate $1/\gamma$. The total convergence rate $\eta$ upon returning to the Poincaré section is determined by the product of $\gamma_i (i=1, 2, 3, \ldots)$ at each location of reflection of the attractor. The Lyapunov exponent

$$
\lambda = -\ln \eta = -\ln \prod_{i=1}^{2(n+1)} \gamma_i
$$

characterizes the rate of convergence on an exponential basis. The negative sign in the definition ensures that positive $\lambda$ is associated with attractors. If $\lambda=0$ then no attractors exist between the double ridge. As mentioned earlier, this Lyapunov exponent based approach to determining the existence and stability of closed orbits connects with more general approaches to determining the existence of LCS in dynamical systems.

To summarize, the mathematical tool developed in this section extends the ray tracing approach used in Maas and Lam and Echeverri et al. by placing the existence of attractors on a rigorous footing. The creation of $d$ for each initial condition $X$ allows less computation for attractor extraction. Coordinate stretching motivated by the WKB approximation also allows us to apply this tool to real ocean data. In Secs. III and IV, we employ the mathematical tool developed above to locate internal wave attractors in idealized double ridge configurations as well as realistic ocean double ridge systems.
the left ($X^*_L$) is on a stable limit cycle and that the right ($X^*_R$) is on an unstable limit cycle (it is in the reverse direction of the characteristic initiated from the left fixed point indicated by the arrow). From our discussion in Sec. II, the stability of the fixed point on the right will change when characteristics are initiated to the right. In fact, the (2,1) attractor comprises these two fixed points. For a leftward propagating characteristics initiated at $X^*_L$, it reflects at the ocean floor twice and returns to $X^*_R$, then propagates to the right. The slope of $d$ at $X^*_R$ indicates that it is the stable fixed point on the Poincaré section for characteristics propagating leftward from the surface. When a characteristic is initiated to the left from $X^*_R$, the direction of energy flow is reversed, and thus the attractor becomes a repeller. The slope of $d$ at $X^*_R$ indicates that it is an unstable fixed point on the Poincaré section for leftward propagating characteristics. Note that in both panels, the arrows on the limit cycles indicate the direction of the flow of energy, and the line types indicate the associated stability type for left initiated characteristics (solid indicating stable limit cycle and dashed indicating unstable limit cycle), whereas the arrows on the ocean surface indicate the corresponding stability of the fixed point.

In Fig. 3 we present the parametric dependence of the distance functions in Fig. 2 on $L$. For both closed orbits, a saddle node bifurcation occurs at the upper limits of $L$ that support attractors. As $L$ decreases, the fixed points separate further. In Fig. 3(a), the distance function $d$ is continuous, yet its derivative is discontinuous at some point between $X=0.5$ and 1. The nonsmoothness of $d$ for the (1,1) attractors is due to a nonsmooth connection between the two Gaussian profiles, as evident in Fig. 2(a). For both figures, we note that $d$ approaches an infinite slope toward the right due to characteristics approaching the critical slope of the right ridge. Although the saddle node bifurcation is similar for both cases, the disappearance of closed orbits is different. Due to the connection between the two fixed points, the (2,1) attractor disappears when $d$ only has one zero (as shown by the marginal case $L=2.35$), and characteristics escape the domain of interest, yet the (1,1) attractor disappears only when $d$ has no zeros (as shown by $L=1.37$).

We keep track of the Lyapunov exponents associated with wave attractors and summarize the parameter space that supports the attraction in Fig. 4 for the initially left-going characteristics). Figure 4(a) shows the Lyapunov exponent $\lambda$ associated with each set of parameters. If no attractor exists, the Lyapunov exponent $\lambda$ is set to 0. We find three major basins in parameter space for the (1,1), (2,1), and (3,1) attractors. The lower limit in $L$ for each basin shows strong attraction due to the orbit hitting a convex critical point [cf. Fig. 4(b)] and characteristics experience strong convergence at this reflection. At the upper limit of $L$ for the (1,1) and (3,1) basins [Figs. 4(c) and 4(e)], the attractor deforms to a form in which both vertical reflections on the supercritical flanks of the ridges are close to the convex supercritical points, and so there may be constructive interference in the wave field. The attractor itself, however, is weak due to the orbit striking at similar slope angles on both ridges, leading to approximately the same amount of focusing/defocusing. The upper limit of the (2,1) attractor [Fig. 4(d)] is different as it corresponds to the closed orbit reflecting back and forth between the two concave critical points and the ocean surface, and so destructive interference may be present here. It is also worth noting that the basin for the (2,1) attractor is wider than the other two basins, and the Lyapunov exponents are also bigger due to the characteristics converging twice upon hitting the supercritical slopes.

**IV. A GEOPHYSICAL EXAMPLE: THE LUZON STRAIT**

One example of a double ridge system in the real ocean is in the Luzon strait at the eastern edge of the South China Sea. The two ridges are almost parallel to each other, running north-south between 19° and 23° north and 121° and 122° east. A bathymetric map of the Luzon ridge system is shown in Fig. 5. Motivated by the shape of the double ridge system, we examine the reflection of internal wave characteristics for the 2D geometry at every latitude in this data set. In investigating the possibility of attractors, we have considered six different stratification profiles inside and nearby the region of interest. The data sets, presented in Fig. 5, are mostly obtained in the spring and summer, at which time the thermocline is about 100 m; one density profile was taken in the winter and thermocline is about 200 m due to the cold surface temperature.
At each latitude, we use the algorithm outlined in Sec. III to search for internal wave attractors. First, the topography is stretched using WKB coordinates so that the inclination of internal wave characteristics is 45°. We note that this is equivalent to propagating characteristics in the physical coordinate, but it generalizes the parametric dependence. The distance functions are then computed at every latitude to find zeros which correspond to the fixed points in Poincaré sections. Since the realistic bathymetry is three-dimensional, to have some confidence in the results of the 2D approach, we seek regions where the ridge system is nominally 2D and wave attractors exist throughout multiple consecutive latitudes.

The nonuniform density stratification plays two important roles in determining the ability of the nondimensional geometry to support internal wave attractors. The strong convergence of characteristics is immediately relaxed upon reflecting back to the surface.

FIG. 4. (Color) Internal wave attractors and Lyapunov exponents in a double Gaussian ridge system. (a) Variation in $\lambda$ with respect to $L$ and $\sigma_e$. Major attractors are [(b) and (c)] (1,1) attractor, (d) (2,1) attractor, and (e) (3,1) attractor. Note the weak convergence of the attractor in (d) due to its close proximity to the concave critical point. Strong convergence of characteristics is immediately relaxed upon reflecting back to the surface.

FIG. 5. (Color) (a) Bathymetric map of the Luzon ridge. The studies focus on the region in the highlighted box. Six sets of density profiles from the marked locations are considered. Cross: stratification at 22° north from the Taiwanese National Data Base Center (TNDBC), valid in summer. Circle: TNDBC data sets at 21.8° north, 121.2° east. Stratification data for both summer and winter are collected at this location. Diamond: Generalized Digital Environmental Model (GDEM) stratification at 21° north, 121.33° east, valid from April–July. Hexagram: CTD data from WISE/NLIWI experiments on July 2007, 20.65° north, 121.3° east. Pentagram: stratification from World Ocean Atlas (WOA) at 20° north, 121° east. (b) Buoyancy frequency at different sites. Marker styles correspond to the sites marked in (a). Note that the profile with open circles corresponds to a winter profile and the profile with closed circles corresponds to summer profile. Profiles have been shifted by 0.02 rad/s for clarity. (c) Average stratification $\bar{N}$ associated with different density profiles.
stratification near the ocean surface occupies a significant portion of the WKB coordinate, making tall ridges short (the top 300 m below the ocean surface occupies about the top 1/3 of the WKB coordinates). Furthermore, a decreased mean stratification stretches the distance $L$ between the two ridges, which is an important factor in supporting wave attractors [as seen in Fig. 4(a)]. Two density profiles examined [TNDBC winter and WISE/NLIWI data, shown in Fig. 5(c)] in this study fail to support wave attractors due to this reason.

A significant portion of the Luzon ridge system cannot support attractors for reasonable density profiles for a number of reasons. North of 22.4° north, the east ridge recedes and becomes subcritical, and immediately south of the region highlighted in Fig. 5(a), the ridges quickly become very short in the WKB coordinate. In the midsection of the double ridge system between 20.5° and 21.5° north, as shown in Fig. 6(a), at 21° north, for example, the ridges are farther apart and the geometry falls short on supporting either a (1,1) or (2,1) attractor. Further south, between 19° and 20.5° north, the west ridge becomes subcritical and no wave attractor can be found.

We find that internal wave attractors are generally supported between 21.8° and 22.4° north. The favorable condition is due to the west ridge breaching the ocean surface, and the two ridges being not too far apart, allowing attractors to exist even with a relatively short supercritical east ridge. The WOA and TNDBC summer (21.8° north, 121.2° east) profiles are marginal. The most robust set of attractors can be found with the TNDBC summer (22° north) and, especially, the GDEM stratification profiles. The analyses for two latitudes based on this profile are shown in Fig. 6. In Fig. 6(b), an attractor is supported by the topography at 22.23° north. At this latitude, only one zero exists in the distance function $d$, corresponding to a closed orbit. To further illustrate the wave attractor supported by the GDEM profile, the limit cycles at different latitudes are plotted in Fig. 7 in both physical coordinates and the WKB stretched coordinates. Together, the limit cycles form a set of surfaces that attract nearby characteristics.

V. DISCUSSIONS AND CONCLUSIONS

We have developed an efficient mathematical and numerical tool to find internal wave attractors between 2D parallel submarine ridges. The development of these tools was motivated by recent thinking concerning the identification of LCS, where the geometric pattern of mixing is extracted based on the Lagrangian ray trajectories. WKB stretching is invoked to deal with nonuniform stratification in the ocean. In the stretched coordinate, wave inclinations are at 45° and attraction occurs only upon each reflection at the ocean floor, so the problem can be reduced to stability analyses of fixed points on Poincaré sections. Using this approach, we identify
the basins in parameter space that support various types of wave attractors in idealized double ridge configurations.

We have successfully applied this approach to the study of wave attractors for realistic topography in the ocean. Based on our understanding of the geometric dependence of attractors, the distance between the two ridges in midsection of the Luzon ridge system falls between (1,1) and (2,1) attractors and the western ridge is not significantly supercritical, undermining the possibilities of wave attractors in these regions. However, a significant northern portion of the Luzon ridge can support (1,1) attractors. By studying several density profiles, we have also found the dependence of wave attractors on stratifications. In general, summer stratification is more favorable in supporting wave attractors than winter stratification due to the change in mean stratification, which effectively results in the changing ridge separation. These results, together with the analyses in Sec. III and the experiments in Echeverri et al., show that we do not need a confined basin to support an attractor. As such, they warrant further attention to tidally forced mixing processes near ocean double ridge systems.

Several open issues still remain to further understand and detect the internal wave attractors in the real ocean. First, we have only employed a geometric approach in two dimensions. To fully capture the energetics of the wave field in the real ocean, we would need to consider the full three-dimensions. To fully capture the energetics of the wave field in the ocean double ridge systems.

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