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Stochastic Lagrangian dynamics for charged flows in the E-F regions of ionosphere

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I. INTRODUCTION

The properties of the terrestrial ionosphere, in response to dynamical processes of the ambient environment, are of great interest for space physics and telecommunication. Because of the presence of the ionosphere, electromagnetic waves can reflect and transmit over long distances. Among these properties, structures of the density of plasma flows are very important. They include estimates of the mean density of plasma at different altitudes, as well as the variabilities of density fluctuations. One challenging region for such estimates is in the lower ionospheric altitudes, which is too low for orbiters and too high for radiosondes to take direct measurements.

In recent years, computer simulations of the earth’s ionosphere have become a prevailing tool to obtain properties of plasma flows in the ionosphere, especially at low altitudes. One particular process of interest is the interaction between neutral and plasma flows due to collision. At low altitudes, ion- and electron-neutral collisions can be significant because of the large density of neutral particles. This process becomes significantly weaker at higher altitudes, where neutral density drops below plasma density. Some of the computer simulation studies primarily focus on the role of neutral winds on the generation of field aligned irregularities (FAIs) driven by charge-neutral collisions.\textsuperscript{1–4} These studies are motivated by middle and upper atmosphere Radar observations on ion density disturbances in mid-latitude ionosphere.\textsuperscript{5} The density disturbances are usually found above 100 km altitude in the post-sunset period during the summer time. They are in the form of quasi-periodic echos and can extend up to 130 km in altitude, with periods of 2–15 min. A possible explanation for these echos is the collision between plasma and polarized neutral wind fields (inertial gravity waves, hereinafter referred to as IGWs).\textsuperscript{6,7} The aforementioned deterministic simulations do support these theoretical work on the generation mechanisms.

However, the environment is inherently stochastic, and random motions of charged particles (ion) in addition to the deterministic mean must be accounted for with statistical measures. One quantity of our interest is the variability of the Lagrangian motion of plasma—from the flow trajectories, we can obtain the flow topology, using Lagrangian coherent structures (LCS).\textsuperscript{8–11} LCS is a recently developed mathematical tool to identify the skeletons of nonlinear chaotic mixing processes. Roughly speaking, attractors (repellers) in LCS correspond to convergences (divergences) in flow, indicating generation of fronts (troughs). We favor a Lagrangian approach, as they provide objective description of the flow topology.\textsuperscript{12} With LCS from ion velocity, we can obtain the mixing template for plasma density.

Consider the ion random motion driven by ambipolar diffusion. In deterministic dynamics for the mean, this coefficient acts to homogenize ion density. In the stochastic differential equation (SDE) set up, this coefficient, manifesting itself in the SDE as a vector Wiener process, drives individual realizations of the motion for ion trajectories. This results in interesting statistics on the transport patterns, henceforth the shape of density fronts, for plasma. As such, the random motion is considered to be driven by ambipolar diffusion. In particular, we focus on the stochastic Lagrangian motion of plasma, whose mean drift are based on resolved deterministic velocity, plus two forms of stochastic forcing, to be explained later. Statistics associated with these random motions of the plasma flow are obtained from Monte-Carlo simulations. We follow prior numerical simulation studies\textsuperscript{13} and use a polarized neutral wind field to drive mean ion motion.

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Because the deterministic mean motion is nonlinear, the statistics associated with the motion of plasma could be highly non-Gaussian. In a previous study on neutral wind, non-Gaussian statistics were observed for stochastic Lagrangian trajectories in polarized velocity fields in an IGW.\(^{13}\) We will explore in this study what similarities and differences in statistics can be obtained for plasma.

This paper is organized as follows. In Sec. II, we discuss the mathematical formulation and numerical set up for the problem. In Sec. III, we show numerical results of various deterministic fields and subsequent stochastic Lagrangian studies. In Sec. IV, we discuss the implications of our results and draw conclusions. The actual equations handled by our numerical models. Because of the observations and subsequent developments of the theory and simulations of polarized field aligned irregularities in the low altitude ionosphere, we focus on such regions with our special interest on the statistical properties. In the atmosphere, mesoscale structures are important for practical applications. In IGW forcing is absent; for recombination rates, we use the reference production rate, and reference loss rate. Note that the for-mulation of the loss term is the same as Yokoyama et al.\(^{14}\) since we are concerned with recombination rate for both E-F regions, instead of E-region alone, as simulated in Yokoyama\(^{5}\) where a quadratic term was used for the dominant ion species \(Fe^+\).

In terms of dimensionless parameters, \(k_B' = k_B T_0 / L B_0 V_0 c\) is the Boltzmann constant, \(\phi' = \phi_0 / L V_0 B_0\) the electrostatic potential, \(\rho(\theta) = M_i(\theta) v_i(\theta) / eV_0 B_0\) the normalized collision frequencies, \(g' = M_i g / eV_0 B_0\) the gravity constant, \(P' = N_0 V_0 / L = 68.182\) the reference production rate, and \(L' = V_0 / L = 6.8182 \times 10^{-5}\) the reference loss rate. Note that the formulation of the loss term is the same as Yokoyama et al.\(^{14}\) since we are concerned with recombination rate for both E-F regions, instead of E-region alone, as simulated in Yokoyama\(^{5}\) where a quadratic term was used for the dominant ion species \(Fe^+\). Equations (1) and (2) are the ion and electron momentum equations, respectively. The first terms on the right hand side of these equations are the ion and electron pressure. The second terms correspond to motion induced by the electric field, given as the electrostatic potential \(-\nabla \phi\). The third terms are drift due to magnetic field, and the fourth terms are due to charge-neutral collision.

In addition, gravitational force is considered for ion. Equation (5) is the transport equation for plasma density. Equation (4) is a simplified form of the Maxwell’s equations when the plasma is quasi-neutral and the magnetic field remains constant.

We choose a horizontal length scale of 44 km for the IGW. Our simulation domain is 44 km \(\times\) 44 km in the horizontal directions and between 80 km and 440 km in the vertical. As a result, we capture the altitudes of the E region and part of the F region where plasma density is at its maximum.

The initial profiles for plasma density and temperature are obtained from the IR2007 model,\(^{15}\) for a case on 25 June 1991 at 20:00 LT, at 35° N, 136° W in geographic latitude and longitude. This case corresponds to the field-aligned irregularities in the midlatitude E-region observed in Yamamoto et al.\(^{5}\) For ion and electron collision frequency, we use Kelley.\(^{16}\) For the production rates, we generate a profile such that the observed ion density remains at equilibrium when IGW forcing is absent; for recombination rates, we use the formula in Huang.\(^{7}\) These profiles, in their respective dimensional units, are shown in Fig. 1.

Figure 1(a) shows the plasma temperature. The black solid curve denotes the ion temperature \(T_i\), and the red dashed curve denotes the electron temperature \(T_e\). Figure 1(b) shows the initial plasma density in log scale. In particular, we can see an E-layer centered around 100 km altitude. Figure 1(c) shows the normalized collision frequencies \(\rho(\theta)\). Figure 1(d) shows the production (black solid curve) and recombination rates (red dashed curve).

The mechanism of preferential generation of polarized electric field (south-west propagation preferred than north-east propagation) has been explained in detail in Yokoyama et al.\(^{5}\) The main reason is the angle between the wave vector and the magnetic field line. For midlatitude northern hemisphere, southwest propagation of IGW leads to non-orthogonal intersection with the magnetic field line, hence more amenable to generation of polarized electric fields. As such, for the neutral perturbation, we assume an idealized IGW with phase speed towards the South. Specifically, the IGW takes the form (in non-dimensional terms)

\[
0 = \nabla \cdot [e n (\nabla \phi + \nabla \phi) + g']
\]
\[ u = u_z + U(y) \sin \phi, \quad v = -U(y) \cos \phi, \]
\[ w = u_m + U(y) \cos \phi, \]  
(5)

where \(x, z\) are the non-dimensional coordinates in the zonal and meridional directions, \(y\) is the vertical coordinate, \(u, v, w\) are the non-dimensional velocities in the zonal, vertical, and meridional directions, \(\phi(t) = 2\pi(-5z - 10y - 25t)\) the phase, and \(u_z, u_m\) are horizontal drift velocities. In this study, we consider \(u_z = u_m = 0\). We assume that IGW only propagates through the E region. The neutral velocity magnitude \(U(y)\) is taken to be 1 at the bottom \(y = 0\) and damps exponentially as \(U(y) = -e^{-\gamma y}\). These choices lead to horizontal wave length of 44 km, vertical wave length of 22 km, wave period of about 10 min, and a vertical scale length of 55 km, starting from 15 ms\(^{-1}\) at 80 km.

Because the momentum equations are linear, the ion and electron velocities can be determined analytically from Eqs. (A1)–(A5). The numerical package uses pseudo-spectral method to deal with periodic boundary conditions, which makes the calculation of the density evolution equation when solving for the density evolution equation highly accurate for our problem. Inside this package, the vertical direction is chosen to be \(y\), for convenient data structure optimized for handling spectral derivatives on the two horizontal directions. We thus adopt this coordinate system as discussed around Eq. (5). For the boundary conditions on the electrostatic potential, we require \(\phi = 0\) at the top and equi-potential along the magnetic field line at the bottom. The density is set to maintain diffusive equilibrium at the top and no flux at the bottom.

For the electrostatic potential equation (4), note that over the range of altitudes between 80 km and 440 km, the scale separation in ion and electron collision frequencies, compounded with the density scale separation, contribute to numerical coefficients over 10 orders of magnitude, which makes the Poisson’s equation very stiff to solve. In order to suppress the scale separation coming from these terms, we solve this equation by scaling up at each altitude with \(\rho_i(y)/n_{\text{init}}(y)\), where \(n_{\text{init}}(y)\) is the initial density profile. The resulting equation is solved using the bi-conjugate gradient discussed in Stiejen and Fokkema.\(^{17}\) Note that our Eulerian model is essentially the same as those developed in prior numerical studies.\(^{1-4}\) The novelty of this paper rests on the subsequent Lagrangian analyses which brings new insight in the mixing process for plasma in the ionosphere.

### B. Lagrangian coherent structures

LCS are distinguished material lines/surfaces that attract/repel local trajectories at the maximal rates.\(^{19,20}\) Recently, it has become conventional that the finite-time Lyapunov exponents (FTLEs) be used to extract LCS. Based on local dynamics of trajectories, FTLE offers an objective description of how nearby trajectories stretch with the background flow. As a result, local material lines that repel nearby trajectories the most are identified. Reciprocally, attractors are found as local maximizers of FTLE when trajectories are integrated in backward-time. In the first development of the theory, FTLE fields do not distinguish large stretching from shear.\(^3\) This difficulty has been resolved in recent developments.\(^9,10\)

The computation of FTLE is relatively well known now, henceforth we only briefly outline the methodology. We compute the plasma displacements starting from initial condition \((x_0, t_0)\) using the ion velocities. Denoting the trajectory \(x(t; x_0, t_0)\), we define the Cauchy-Green strain tensor field as

\[ M'(x_0) = \left[ \frac{\partial x(t; x_0, t_0)}{\partial x_0} \right]^T \left[ \frac{\partial x(t; x_0, t_0)}{\partial x_0} \right], \]

where \(\frac{\partial x/\partial x_0}^T\) is the transpose of the deformation gradient tensor \(\partial x/\partial x_0\). The FTLE field, \(\text{FTLE}'_{i0}(x_0)\), is then defined as the scalar field that associates with each initial position \(x_0\) the maximal rate of stretching

\[ \text{FTLE}'_{i0}(x_0) = \frac{1}{2(t - t_0)} \ln \lambda_{\text{max}}(M), \]

with \(\lambda_{\text{max}}(M)\) denoting the maximum eigenvalue of \(M\). Because the FTLE field measures the largest separation among nearby trajectories, in forward time, highlighters of FTLE indicate repellers and in backward time, highlighters of FTLE indicate attractors. It is the backward-time FTLE.

---

**FIG. 1.** Initial profiles in dimensional units. (a) Plasma temperature. Black solid curve: ion temperature. Red dashed curve: electron temperature. (b) Ion density. (c) Normalized collision frequency. Black solid curve: ion. Red dashed curve: electron. (d) Reaction parameters. Black solid curve: production rate, coordinate at bottom of figure. Red dashed curve: recombination rate, coordinate at top of figure. Vertical axes are the same in all panels.
that is of our most interest as they indicate regions prone to
the development of plasma fronts.

III. NUMERICAL RESULTS

A. Initialization at equilibrium

Without IGW and in equilibrium state, Eqs. (A3) and
(A4) reduce to

\[
\frac{k_B (\sin^2 I + \rho_i^2)}{\rho_e (1 + \rho_e^2)} (nT_e)_y - \frac{k_B (\sin^2 I + \rho_i^2)}{\rho_i (1 + \rho_i^2)} \frac{g (\sin^2 I + \rho_i^2)}{\rho_i (1 + \rho_i^2)} n = 0,
\]

\[
= \frac{\phi_i (\sin^2 I + \rho_i^2)}{\rho_i (1 + \rho_i^2)} n \phi_y + \frac{\phi_i (\sin^2 I + \rho_i^2)}{\rho_i (1 + \rho_i^2)} n \phi_y.
\]

Eliminating \( \phi_i \phi_y \) from Eq. (6) and after some manipula-
tions, one obtains

\[
-L' n - P' p = \left[ \frac{g (\sin^2 I + \rho_i^2)}{\rho_i (1 + \rho_i^2)} n \right]_y + \left[ \frac{\phi_i (\sin^2 I + \rho_i^2)}{\rho_i (1 + \rho_i^2)} n \phi_y \right]_y + \left[ \frac{k_B (\sin^2 I + \rho_i^2)}{\rho_i (1 + \rho_i^2)} (nT_e)_y \right]_y.
\]

Once Eq. (8) is solved, one can use it to solve for \( \phi_i \phi_y \) in
Eq. (6), as the equilibrium electrostatic potential profile. Our
production and loss rates are stronger than those used in
Yokoyama,\(^4\) but the derived production rate is consistent
with Huang,\(^2\) which is near their maximum production rate of
300 cm\(^{-3}\) s\(^{-1}\).

To further reduce the problem, turn off production and
loss, Eqs. (6) and (7), become

\[
k_B' (nT_e)_y = \phi_i' n \phi_y, \quad n g' + n \phi'_y + k_B' (nT_e)_y = 0,
\]

which leads to the equation for diffusive equilibrium\(^24\)

\[
\frac{\partial k_B' n T_e}{\partial y} + g' n = 0.
\]

B. Deterministic dynamics

We perturbed the neutral wind by IGW. The undulation of
IGW and density-temperature stratification of plasma immedi-
ately set up a current density divergence, shown in Fig. 2(a) at
\( t = 0 \). Here, the divergence is shown scaled up by \( \rho_e / n_{init} \). The
solved electrostatic potential is shown in Fig. 2(b). As seen,
the undulation is dominant at the bottom near the E layer, where
plasma-neutral collision is the highest and IGW is strong. Eight
horizontal wave periods are shown to place the horizontal and
vertical coordinates in scale.

For the time-evolution of density, we employ explicit
Runge-Kutta-Wray method already implemented in DIABLO.\(^18\)
The time integration is carried out with nonlinear advection and
production terms explicit, yet diffusion and loss terms fully
implicit. This helps on the numerical stability of the code. We
also set an automated selection of the time step based on the
Courant condition.

We show time-evolution of ion density, velocity, and
electrostatic potential in the simulation, in Figs. 3 and 4. In
these two figures, the left panel (a) shows the dimensionless
ion density in log scale, so the density perturbation is visible
under strong vertical stratification. To be precise, the quanti-
ty being plotted is \( \log_{10} (n) \), where again, \( n \) is the dimen-
sionless ion density. The red region between 110 km and
130 km altitudes corresponds to the E region, where the
dimensional ion density is on the order of \( 10^3 - 10^4 \) cm\(^{-3}\). As
seen, the polarized neutral wind has driven the E region
to take polarized structures as well. The black iso-contours of
density help to visualize the density perturbations. Panels
(b)-(d) show the zonal, vertical, and meridional velocity in
dimensional units, respectively. The zonal velocity signifi-
cantly damps at around 130 km, whereas the vertical and me-
ridional velocities keep mapped to high altitudes, since they

FIG. 2. (a) Current density divergence scaled by \( \rho_e / n_{init} \). (b) The resulting electrostatic potential. Eight wave periods are shown, \( y \) is
the vertical coordinate (altitude), and \( z \) is the meridional direction.
are in the same plane as the magnetic field. The right panel (e) shows electrostatic potential, where structures are predominantly at the bottom. The electric field between 90 km and 130 km are plotted as well, showing similar polarization structure. Fig. 3 shows the field values a quarter wave period, whereas Fig. 4 shows those at three quarters of a wave period.

These results corroborate with those discussed in Yokoyama, where the electric fields are at lower altitudes around sharp density gradients for a Southward propagating IGW, yet plasma velocity propagates higher. (cf. Fig. 5 in their paper.) Our results of density perturbation and electrostatic potential at the E region do not look as significant as the case presented in Yokoyama, because in the observed profile, the vertical density stratification in our case is very strong above the E region (as compared to 0) and the E region itself is not as significant (on the order of \(10^3\) cm\(^{-3}\) as compared to \(10^5\) cm\(^{-3}\)).

C. Stochastic Lagrangian dynamics

With the time history of the plasma velocity, we can perform Lagrangian analyses on the density to study the formation of charged fronts. Specifically, we consider the formation of LCS under random perturbation. As mentioned before, these random perturbations are due to ambipolar ion diffusion, which is inhomogeneous and anisotropic in the direction orthogonal to the magnetic field. We use the Stratonovich formulation of the SDE, which is more suitable for such an inhomogeneous diffusion coefficient. When the anisotropy is only considered diagonal,

\[
dx = v_i(x, t)\, dt + \sqrt{2D(x)} \circ dW(t),
\]

where

\[
dx = v_i(x, t)\, dt + \sqrt{2D(x)} \circ dW(t),
\]

are the stochastic terms, and \(D(x)\) is the diffusion coefficient that may vary with the position. The Stratonovich formulation is given by

\[
dx = \frac{v_i(x, t)}{\sqrt{D(x)}}\, dt + \sqrt{D(x)} \circ dW(t).
\]

This formulation is more suitable for our case, where the diffusion coefficient \(D(x)\) is inhomogeneous.

\[
dx = \frac{v_i(x, t)}{\sqrt{D(x)}}\, dt + \sqrt{D(x)} \circ dW(t).
\]

FIG. 3. (a) Normalized density in log scale. (b) Zonal velocity. (c) Vertical velocity. (d) Meridional velocity. (e) Electrostatic potential. (b)–(e) are in dimensional units. Snapshot is at \(t = T/4\), a quarter wave period.

FIG. 4. (a) Normalized density in log scale. (b) Zonal velocity. (c) Vertical velocity. (d) Meridional velocity. (e) Electrostatic potential. (b)–(e) are in dimensional units. Snapshot is at \(t = 3T/4\), three quarters of a wave period.

FIG. 5. Trajectory comparison for initial conditions of plasma parcels released at \(x = 1\) km, \(z = 19\) km and \(y\) between 80 and 220 km. The dots denote the end position of trajectories after integration for a wave period. (a) Deterministic. The three red dots denote trajectories started at \(y = 100, 160, \) and 220 km, respectively. (b) Random case 1 with diagonal diffusivity. The three red layers denote trajectories started at \(y = 100, 160, \) and 220 km, respectively. (c) Random case 2 with field aligned diffusivity. The three red layers denote trajectories started at \(y = 100, 160, \) and 220 km, respectively. Note that the \(z\) scale is much larger than the \(x\) scale, thus the spread is more in \(z\) as compared to \(x\).
where $D$ is the spatially inhomogeneous and diagonally anisotropic ambipolar diffusion coefficient, and $W(t)$ a vector Wiener process. The mean trajectories can be solved by either Monte-Carlo methods or by Fokker-Planck equations.\textsuperscript{21,22} We nevertheless prefer the use of Monte-Carlo here since the ion velocity field is more complicated than the idealized IGW discussed before, hence it is more costly to solve for the Fokker-Planck equations of the evolution of small concentrated probabilities at all initial locations.

We use the Euler-Heun method to solve Eq. (11).\textsuperscript{23} Specifically, we iterate the sample path by

$$ x_{n+1} = x_n + [(1-\theta)v_x(t_n, x_n) + \theta v_x(t_{n+1}, x_n^*)]\Delta t + \frac{1}{2} \left[ \sqrt{2D(x_n^*)} + \sqrt{2D(x_{n+1}^*)}\right] \Delta W_n, \quad (12) $$

where $\theta \in [0,1]$ is a parameter chosen to be $1/2$, $x_n^* = x_n + v_x(t_n, x_n)\Delta t$ a deterministic predictor, and $x_{n+1}^*$ a stochastic predictor.

Two forms of diffusivity are considered. Both of them are introduced here for easy comparison. Observe that in Eq. (A5), the last row related to ion pressure (in the original form) can be expressed as follows:

$$ \mathbf{v} \cdot [D\nabla(nTi)] = \mathbf{v} \cdot \begin{pmatrix} \frac{k_B\rho TI_i}{1 + \rho_i^2} & \frac{k_B\cos IT_i}{1 + \rho_i^2} & \frac{k_B\sin IT_i}{1 + \rho_i^2} \\ -\frac{k_B\cos IT_i}{1 + \rho_i^2} & \frac{k_B(\sin^2 I + \rho_i^2)}{\rho_i(1 + \rho_i^2)}T_i & -\frac{k_B\cos I\sin IT_i}{\rho_i(1 + \rho_i^2)} \\ -\frac{k_B\sin IT_i}{1 + \rho_i^2} & -\frac{k_B\cos I\sin IT_i}{\rho_i(1 + \rho_i^2)} & \frac{k_B(\cos^2 I + \rho_i^2)}{\rho_i(1 + \rho_i^2)}T_i \end{pmatrix} \begin{pmatrix} n_i \\ n_y \\ n_z \end{pmatrix} + \begin{pmatrix} \frac{k_B\cos I}{1 + \rho_i^2}T_{i\gamma n} \\ -\frac{k_B\cos I}{\rho_i(1 + \rho_i^2)}T_{i\gamma n} \\ -\frac{k_B\cos I}{\rho_i(1 + \rho_i^2)}T_{i\gamma n} \end{pmatrix}. \quad (13) $$

Divergence of the last vector is part of drift for $n$, and the diffusion is characterized by the diffusion tensor given as the $3 \times 3$ matrix in Eq. (13). We further simplify the diffusion matrix by expanding the divergence and noting that the $n_y$ and $n_z$ terms cancel (cf. Eq. (A5) after all the cancellations). This leads to the diffusion matrix to take the form

$$ \begin{pmatrix} \frac{k_B\rho IT_i}{1 + \rho_i^2} & 0 & 0 \\ 0 & \frac{k_B(\sin^2 I + \rho_i^2)}{\rho_i(1 + \rho_i^2)}T_i & -\frac{k_B\cos I}{\rho_i(1 + \rho_i^2)}T_i \\ 0 & -\frac{k_B\cos I}{\rho_i(1 + \rho_i^2)}T_i & \frac{k_B(\cos^2 I + \rho_i^2)}{\rho_i(1 + \rho_i^2)}T_i \end{pmatrix}. $$

The first form of diffusion (hereinafter referred to as the "diagonal diffusion case") is based on the diagonal terms in the diffusion tensor, hence $D = [k_B\rho IT_i/(1 + \rho_i^2), k_B(\sin^2 I + \rho_i^2)/\rho_i(1 + \rho_i^2)]^T$. Note that when $\rho_i \gg 1$, this coefficient converges to $k_B^0 T_i/\rho_i$. In this form of diffusion, we have assumed that the inhomogeneity is only in the vertical direction and anisotropy is mainly in $x$, since $I = 45^\circ$ and the vertical and meridional diffusivities are the same. This form of diffusion does not fully capture the anisotropy aligned with the magnetic field, yet it provides some insights on the random process in much simpler form.

The second form of diffusion (hereinafter referred to as the "field-aligned diffusion case") considers the anisotropy alignment along the field line. Because of the symmetry in the simplified diffusion matrix, we can diagonalize diffusion by a simple rotation of $45^\circ$. The resulting diagonal diffusivity, in zonal, $B_\perp$ and $B_\parallel$ directions, are given by $D = [k_B^0\rho IT_i/(1 + \rho_i^2), k_B^0 T_i/\rho_i, k_B^0\rho IT_i/(1 + \rho_i^2)]^T$. In practice, $W$ is generated diagonally based on $D$, and then rotated to align with the magnetic field.

In both forms of stochasticity, 2000 realizations are computed at each spatial initial condition. The ambipolar diffusion is scale dependent and becomes very large at high altitudes, because it is inversely proportional to the collision frequency, which is very small there. We also only focus on the bottom of the domain for Lagrangian analyses. Due to limited randomness at the bottom of the domain, plasma motion is quite similar to deterministic ones, yet plasma trajectories at higher altitudes have more spreading. A comparison of realizations of these trajectories are shown in Fig. 5.

In Fig. 5(a), deterministic trajectories are released at $x = 1$ km, $z = 19$ km and $y$ between 80 and 220 km, at 1 km intervals. The end positions of these trajectories after integration of one wave period are shown. As reference, the three red dots denote those positions for trajectories initially released at $y = 100, 160,$ and 220 km, respectively. In Fig. 5(b), we show the end positions of all stochastic realizations at the same release points. The stochasticity is subject to the first form, with diagonal diffusivity from the diffusion tensor. The three red layers are end positions of realizations released at the same locations of the three red dots in the deterministic case. Note that the length scale in $z$ is much larger than the scale in $x$, indicating that the spread is more in the zonal direction than the meridional direction. In Fig. 5(c), we show stochastic trajectories based on the field aligned diffusivity. The three red layers are again referenced to show how stochastic trajectories spread. In this case, the $B_\parallel$ and $B_\perp$ anisotropy can be clearly seen, as the red layer of points get more aligned in a predominant direction. Note also the variability in diffusion at the bottom and the top in the two stochastic cases. At the bottom, when diffusivity is small, the cluster of trajectories is very
compact. At the top, because of the decrease in $\rho_i$, and the increase in $D$, the cluster spreads much wider.

Based on random trajectories, one could obtain statistics of the position process at every single initial condition. Such a plot reveals the variability that a scalar experiences as it traverses through the domain. The first four integer moments of the position process are shown in Figs. 6 and 7. In both figures, the top, center, and bottom rows show the statistics in $x$, $z$, and $y$ directions, respectively. The first column on the left shows the mean displacements in dimensional units, of a vertical layer of initial conditions located at a constant $x = 1$ km plane. They indicate how far the average trajectory is from their initial conditions. As seen, polarized mean displacement fields are present for both cases, in all three directions. The integration time is a wave period, and the variability in displacements vary on the order of 1 km. Note especially that there is a biased drift in both the meridional $z$ direction and vertical $y$ direction, indicating the net effect of ion velocity on plasma motion. Similar biased drift in the $y, z$ directions is also observed in deterministic dynamics (not shown). The second column shows the standard deviation in dimensional units. Standard deviation increases with height in all directions, because of the decrease in collision frequency. In both cases, the polarization structure in standard deviation is most visible in the vertical direction in the bottom row. The third column shows the skewness. This measure highlights the asymmetry of the probability density associated with a scalar. For a perfect Gaussian process, the skewness would be 0. As seen, in panel (a3) of both figures, the skewness is around 0, indicating that the process in $x$ is Gaussian. In panel (c3) of both figures, polarized structure is seen for the skewness in the vertical $y$ direction, yet in panel (b3), for the diagonal diffusion case Fig. 6, skewness is around 0 with a weak polarized drift in both the $x$ and $y$ directions.
structure, whereas for the field aligned diffusion case Fig. 7, skewness is non trivial and polarized at higher altitudes. This is due to the anisotropy introduced in the y–z direction. In the first case, the diagonal diffusivities are the same, hence diffusion is isotropic as far as the y–z plane is concerned. In the second case, the diffusivities aligned with $B_\parallel$ and $B_\perp$ differ significantly at higher altitudes, hence bringing in the anisotropy. At the bottom of the domain when the normalized collision frequency is large, the diffusion coefficients converge to the isotropic ambipolar diffusion coefficient, hence we do not see as much polarized structures there for skewness. Finally, the right column shows the kurtosis. For a perfect Gaussian process, the kurtosis is 3. In both cases, the kurtosis in x is 3. Larger than 3 kurtosis can be found above 120 km and take polarized structures in the y direction, yet for kurtosis in z, it is almost 3 for diagonal diffusion case, and there is a weak signature of larger than 3 kurtosis at high altitude for the field aligned diffusion case. If kurtosis is greater than 3, the probability density structures are more peaked, yet it is more flat with kurtosis less than 3. From both Figs. 6 and 7, we find polarized structures of nonzero skewness, and larger than 3 kurtosis. This indicates that in those regions, there is higher probability for stochastic trajectories to stay near the mean. The variance plotted in dimensional units, on the other hand, reveals that the spread of trajectories are not very far from the mean, which in addition suggests that the mean dynamics is quite important here, at least at finite time over one wave period.

We consider the Lagrangian integration time to be one wave period, since for Lagrangian analyses in finite time, the best integration time is chosen to be at an intermediate time scale. To further confirm the importance of mean dynamics for our particular case at this time scale, we generate $10^6$ realizations of stochastic trajectories at $y = 220$ km, the highest altitude of our stochastic simulations, where ambipolar diffusion is the strongest, for a single initial condition. The initial condition is chosen where nontrivial skewness and kurtosis in y are found. For this initial condition, we integrate trajectories over one wave period. We generate the probability density function (pdf) by counting the number of trajectories in each grid cell, and compare the highest density regions and the mean trajectory. This is done for both stochastic cases, shown in Fig. 8. In Fig. 8(a), we show the pdf for displacement in all three directions for the diagonal diffusion case, whereas in Fig. 8(b), we show those for the field aligned diffusion case. In both panels, the blue solid curves are the pdf for x displacement, the black dashed curves are the pdf for z displacement, and the red dash-dotted curves are the pdf for y displacement. The horizontal coordinate of the colored crosses mark the mean displacements. Their vertical coordinates are chosen so that the locations can be easily compared to the pdf peaks. As seen, the $x$ pdf is Gaussian, whereas there are some slight asymmetry in the $y$ and $z$ pdf’s. As a result, the mean displacement locations for $y$ and $z$ do not match precisely to the peaks. However, they stay fairly close to the peaks, further suggesting that the mean dynamics do still play important roles in our particular problem. As the integration time for stochastic trajectories increases (not shown), we find that the skewness and kurtosis continue to grow, highlighting more non-Gaussianity. However, they are less relevant to the LCS studies.

Using the mean trajectories, we generate attracting Lagrangian coherent structures based on the finite-time Lyapunov exponents. The results are shown in Fig. 9. In Fig. 9(a), the deterministic attractors are revealed. The LCS take polarized structures. At the bottom of the domain, the attractors are at the strongest, in the sense that they attract nearby trajectories the most. Because of the significant damping in ion velocity above 140 km, the attracting structures become very weak. As a comparison, we show those attractors for both forms of randomness in Figs. 9(b) and 9(c). The diagonal diffusion case is seen in Fig. 9(b), and the field-aligned case is in Fig. 9(c). For both cases, the well formed LCS are seen below 120 km. In this region, stochasticity is very weak, and so the mean dynamics behave strongly similar to deterministic dynamics. Henceforth, the structures are quite similar to the deterministic structures. However, further up, beyond 120 km, the advection speed is weak as compared to the randomness induced by diffusion. Henceforth, the image becomes significantly fuzzy, and no predominant structures are found.

Finally, to show what the density field would look like subject to random motion, we create a bin for number count of particles inside each box of domain, and plot particle

![FIG. 8. Probability density function compared to the mean trajectory. The initial condition is chosen at $x = 0$ km, $y = 43$ km, and $z = 220$ km, where nontrivial skewness and kurtosis are present. (a) Diagonal diffusion. (b) Field aligned diffusion. The blue solid curve is the pdf for $x$ displacement. The black dashed curve is the pdf for $z$ displacement, and the red dash-dotted curve is the pdf for $y$ displacement. The three crosses correspond to the mean value for $x$, $y$, and $z$, respectively. Their vertical coordinates are chosen so the location and the peak density are easily comparable.](image)
density in Fig. 10. Fig. 10(a) shows the number density after one wave cycle for the diagonal diffusion case, and Fig. 10(b) shows those for the field-aligned diffusion case. Note that this density is initiated with 1000 realizations uniformly spaced in each cell of $1 \text{ km} \times 1 \text{ km} \times 1 \text{ km}$, hence effectively it represents uniform density. After one wave cycle, the number density for the stochastic cases becomes aligned with the polarized velocity fields at the bottom, and in agreement with the attractors revealed above. The increase in number density is due to nonlinear attractors in the charged flow dynamical system, which accumulate nearby trajectories of plasma parcels. Above 120 km, diffusion is strong, and for both cases the density field from stochastic trajectories appears to be quite random, consistent with the randomness in the stochastic attractors.

### IV. DISCUSSIONS AND CONCLUSIONS

We studied the statistical properties of the motion of plasma flows in the ionosphere by first formulating a 3D numerical solver for charged flows in the regions of interest. The analyses is primarily based on stochastic Lagrangian motion of plasma particles via solutions to stochastic ordinary differential equations, whose randomness is anisotropic, inhomogeneous, and dependent on the ion-neutral collision frequency, as well as the magnetic field line.

Based on the Eulerian simulation of ion velocity, we find that polarized electric field develops due to excitation of polarized neutral gravity waves. Accordingly, the ion velocity field is also polarized. This nonlinear motion gives rise to non-Gaussian statistics associated with particle displacement. In particular, in the vertical direction, we observe that the structures for standard deviation, skewness, and kurtosis are highly polarized, for both cases of isotropic and anisotropic randomness. The meridional displacement sees more polarization effects with anisotropic diffusion. The Lagrangian coherent structures, in terms of attracting structures reflected by backward-time FTLE fields, acquire polarization effects both in deterministic and stochastic flows. Due to low diffusion below 120 km, the stochastic LCS behave similarly as the deterministic LCS, yet higher up, stochasticity makes the FTLE field more random.

It is important to note that even though we have obtained statistics based on ion flow driven by nonlinear neutral motion, the mean background flow is set to be quiescent. It is important for neutral shear at the E layer to set up the high density E region during night time. Such effects are not studied here. In addition, the background neutral flow is taken to be idealized gravity wave. It will be interesting to see how stochastic Lagrangian motion of ion flow would form coherent structures due to a breakdown of the neutral wind fields. This will be done by actual simulation of the Navier-Stokes equations of the neutral field, and subsequently using those fields to drive ion motion. Such studies are currently underway.

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APPENDIX: EXPANDED EQUATIONS

Here, we include various expanded versions of the equations that are solved numerically. First, substituting Eqs. (1) and (2) into Eq. (4), the ion/electron velocities are

\[
\begin{pmatrix}
\rho_x \\
\rho_y \\
\rho_z
\end{pmatrix}
= \frac{\rho_i^2}{1 + \rho_i^2} \begin{pmatrix}
\cos I \rho_i & \sin I \rho_i \\
-\cos I \rho_i & \cos I \sin I \\
-\sin I \rho_i & -\cos I \sin I
\end{pmatrix} \begin{pmatrix}
\frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}
\end{pmatrix} \begin{pmatrix}
\frac{\phi_e}{\rho_i} \\
\frac{\phi_i}{\rho_i} \\
\frac{\phi_e}{\rho_i} \\
\frac{\phi_i}{\rho_i} \\
\frac{\phi_e}{\rho_i} \\
\frac{\phi_i}{\rho_i} \\
\end{pmatrix},
\]

(A1)

\[
\begin{pmatrix}
\rho_x \\
\rho_y \\
\rho_z
\end{pmatrix}
= \frac{\rho_e^2}{1 + \rho_e^2} \begin{pmatrix}
-\cos I \rho_e & -\sin I \rho_e \\
\cos I \rho_e & -\cos I \sin I \\
\sin I \rho_e & -\cos I \sin I
\end{pmatrix} \begin{pmatrix}
\frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}
\end{pmatrix} \begin{pmatrix}
\frac{\phi_e}{\rho_e} \\
\frac{\phi_i}{\rho_e} \\
\frac{\phi_e}{\rho_e} \\
\frac{\phi_i}{\rho_e} \\
\frac{\phi_e}{\rho_e} \\
\frac{\phi_i}{\rho_e} \\
\end{pmatrix},
\]

(A2)

where we have assumed that the magnetic field vector is in the y–z plane.

Plugging in the expressions (A1) and (A2) into the last equation of (4), we can solve for the electrostatic potential. The expanded version of this equation is

\[
\begin{align*}
\left\{ \frac{1}{1 + \rho_i^2} \left[ \rho_i^2 (\nabla u_{n_3}) - k_B^3 \rho_i \partial (n_{T_e}) \right] + \cos I \left( \rho_i \nabla u_{n_3} - k_B^3 \partial (n_{T_e}) - g_e n \right) \right. \\
+ \left. \sin I \left( \rho_i \nabla u_{n_3} - k_B^3 \partial (n_{T_e}) \right) \right\}_x \\
+ \left\{ \frac{1}{1 + \rho_i^2} \left[ \rho_i^2 (\nabla u_{n_3}) - k_B^3 \rho_i \partial (n_{T_e}) \right] + \cos I \left( \rho_i \nabla u_{n_3} - k_B^3 \partial (n_{T_e}) - g_e n \right) \right. \\
+ \left. \sin I \left( \rho_i \nabla u_{n_3} - k_B^3 \partial (n_{T_e}) \right) \right\}_y \\
- \left\{ \frac{1}{1 + \rho_i^2} \left[ \rho_i^2 (\nabla u_{n_3}) - k_B^3 \rho_i \partial (n_{T_e}) \right] - \cos I \left( \rho_i \nabla u_{n_3} - k_B^3 \partial (n_{T_e}) \right) \right. \\
- \left. \sin I \left( \rho_i \nabla u_{n_3} - k_B^3 \partial (n_{T_e}) \right) \right\}_z
\end{align*}
\]

Finally, we show the expanded equation for the evolution of ion density. Substituting the velocity expressions into Eq. (3), one obtains

The left hand side of the equal sign of Eq. (A3) corresponds to the divergence of charged currents that forces the electric potential to balance on the right hand side.
\[
\frac{\partial n}{\partial t} - P + L n = - \nabla \cdot (n \mathbf{v}), \quad (A4)
\]
where the transport term is (after rearranging and considering \( \rho_i, T_i \) are only functions of \( y \))

\[
\nabla \cdot (n \mathbf{v}) = - \frac{\phi' \rho_i}{1 + \rho_i^2} (n \phi_i)_x + \left( \frac{\phi' \cos I}{1 + \rho_i^2} (n \phi_i)_y + \frac{\phi' \sin I}{1 + \rho_i^2} (n \phi_i)_z \right) \cdot \frac{\phi' \cos I}{1 + \rho_i^2} (n \phi_i)_y - \frac{\phi' (\sin^2 I + \rho_i^2)}{1 + \rho_i^2} (n \phi_i)_y
\]

\[
+ \frac{\phi' \sin I \cos I}{\rho_i(1 + \rho_i^2)} (n \phi_i)_y - \frac{\phi' \sin I \cos I}{\rho_i(1 + \rho_i^2)} (n \phi_i)_y + \frac{\phi' \sin I \cos I}{\rho_i(1 + \rho_i^2)} (n \phi_i)_y
\]

\[
- \frac{\rho_i \cos I}{1 + \rho_i^2} (n \mu_{n1})_y - \frac{\rho_i \sin I}{1 + \rho_i^2} (n \mu_{n1})_z + \frac{\rho_i \cos I}{1 + \rho_i^2} (n \mu_{n2})_y - \frac{\rho_i \cos I}{1 + \rho_i^2} (n \mu_{n2})_y - \frac{\rho_i \sin I}{1 + \rho_i^2} (n \mu_{n3})_x + \frac{\rho_i \sin I}{1 + \rho_i^2} (n \mu_{n3})_x
\]

\[
- \left[ \frac{k' \rho_i \cos^2 I + \rho_i^2}{\rho_i(1 + \rho_i^2)} (n T_i)_y \right] - \frac{k' \rho_i \cos I}{\rho_i(1 + \rho_i^2)} (n T_i)_y - \frac{k' \rho_i \cos I}{\rho_i(1 + \rho_i^2)} (n T_i)_y + \frac{k' \rho_i \cos I}{\rho_i(1 + \rho_i^2)} (n T_i)_y + \frac{k' \rho_i \cos I}{\rho_i(1 + \rho_i^2)} (n T_i)_y.
\]

(A5)

The first two rows on the right hand side of Eq. (A5) are due to the electric field; rows 3 and 4 (except for the last term) are due to ion-neutral collision and gravity; the last term in row 4 and the entire row 5 are due to inhomogeneous diffusion of ion. Rows 1-4 (except for the last term) are dealt with using explicit numerical methods, whereas the diffusion terms are advanced implicitly.