MAT 272  
Practice Problems for Final Review Chapters 12, 13, 14, 15

1.  \( f(x, y) = x^2 + y - 2 \). Sketch \( f(1, y) \) and \( f(x, -1) \).

2. Find the equation of the plane containing the line \( x = 4 - t, y = 2 + 3t, z = -1 - 2t \) and the point \((1, 1, 1)\).

3. a) Find the equation of a straight line through the point \((3, 7, -2)\) and parallel to the plane \(4x - 6y + 2z = 3\).
   b) Find the distance from the line to the plane.

4. Find the equation of the plane containing the line \( x = 4 - t, y = 2 + 3t, z = -1 - 2t \) and the point \((1, 1, 1)\).

5. \( f(x, y) = 2 + \cos(x - y) \), find the equation of a contour at 2.5 and sketch its graph.

6. For the function \( z = 3 - x^2 + y \), sketch the intersection with the xy-plane, the xz plane, and the yz plane. Label the axes and the intercepts.

7. Find the equation of the plane which passes through the points \((3, 2, -1)\), \((1, -1, 3)\) and \((3, 2, 4)\).

8. Find the area of the triangle defined by the three points in problem 7 and find the measures of the angles of the triangles.

9. The motion of a particle is given by the vector function \( \mathbf{r}(t) = (2t^{3/2}, \cos 2t, \sin 2t) \), \( 0 \leq t \leq 1 \).
   a) Find its velocity and acceleration at \( t = 1/2 \)
   b) Calculate the length of the trajectory traced by the particle from \( t = 0 \) to \( t = 1 \).

10. \( f(x, y) = x^3 - 3x + y^3 - 9y^2 + 24y \) has a relative maximum, a relative minimum, and two saddle points. Find the x and y coordinates of these points.

11. \( f(x, y) = x^3 + xy^2 + y^2 + 3x^2 \) \(-3 \leq x \leq 3\), \(-3 \leq y \leq 3\)
   Determine the coordinates of the relative and absolute maxima, relative and absolute minima, and saddle points.

12. The airline's luggage restrictions state that the sum of length width and height of a piece of luggage cannot exceed 147 cm. Find the dimensions of a rectangular piece of luggage that maximizes the volume of the luggage.

13. Find an equation for the tangent plane to \( f(x, y) = x^2y - x y^3 + 2 \) at the point \((3, 2)\).

14. Find the derivative of \( f(x, y) = xy^2 \) at \((1, 3)\) in the direction toward \((4, 5)\).

15. For the function \( f(x, y) = 8 - 4x^2 - 2y^2 \), find the direction of the steepest grade at \((1, 1, 2)\).

16. If the temperature is given by \( T(x, y, z) = 3x^2 - 5y^2 + 2z^2 \) and you are located at \((0, 3, 0.2, 0.5)\) and want to get cool as soon as possible, in which direction should you head?

17. Consider the function \( f(x, y, z) = x^3 + y^3 - z \).
   a) find the rate of change of this function at the point \((1, 1, 2)\) in the direction toward the point \((-2, 3, 5)\).
   b) At the point \((1, 1, 2)\) find a direction (not a zero vector) in which the directional derivative is zero.
18. Find $\frac{\partial z}{\partial x}$ if $x^2 + \sin^2 y + 2z = x(1 - z^2)$.

19. A cylindrical piece of steel is initially 8 inches long and has a diameter of 8 inches. During heat treatment the length and diameter each increase by 0.1 in. Use differentials to find the approximate increase in the volume.

20. Find the rate of change with respect to time of the hypotenuse of a variable right triangle at the instant when the two sides are 7 and 10 inches if the first side is increasing at the rate of 6 in./min. and the second is increasing at the rate of 4 in./min.

21. The contour diagram below shows the contours of some function, $f(x, y)$. Use it to estimate the following

   a) $f_x(0, 0.7)$  
   b) $f_y(0, 0.7)$  
   c) $D_\vec{v}f(-1, 0)$ if $\vec{v} = (1, 1)$

22. A soft drink manufacturer wants to design an aluminum can in the shape of a right circular cylinder to hold a given volume, V. If the objective is to minimize the amount of aluminum needed (top, sides and bottom), use the method of Lagrange multipliers to find what dimensions should be used.

23. A space probe in the shape of the ellipsoid $4x^2 + y^2 + 4z^2 = 16$ enters the earth’s atmosphere and its surface begins to heat. After one hour, the temperature at the point $(x, y, z)$ on the probe’s surface is $T(x, y, z) = 8x^2 + 4yz - 16z + 600$. Use the method of Lagrange multipliers to find the hottest point on the surface.

24. Write integral expressions that are equivalent to the following, but with the order of integration reversed.

   a) $\int_{-2}^{0} \int_{0}^{2+x} f(x, y) \, dy \, dx + \int_{0}^{2} \int_{0}^{2-x} f(x, y) \, dy \, dx$  
   b) $\int_{1}^{3} \int_{x}^{3x} f(x, y) \, dy \, dx$

25. A solid is bounded by the plane $z + y = 4$, $y = 0$, $x = 0$ and the parabolic cylinder $z = x^2$. Put appropriate limits on the triple integrals shown in the order given and evaluate the volume.

   a) $\iiint 1 \, dz \, dy \, dx$  
   b) $\iiint 1 \, dy \, dz \, dx$
26. Evaluate $\iiint_E (x^2 + y^2) \, dV$ where $E$ is the ice cream cone-shaped solid bounded by the surfaces $z = \sqrt{2x^2 + 2y^2}$ and $x^2 + y^2 + z^2 = 27$.

27. Find the volume of the solid between the spheres $x^2 + y^2 + z^2 = 16$ and $x^2 + y^2 + z^2 = 4$ bounded by the cone $z = \sqrt{x^2 + y^2}$.

28. Evaluate $\iiint_E x \, dV$ where $E$ is the solid bounded by the planes $x = 0$, $z = 0$, $x + 2y + z = 2$.

Possible Answers:

1. $f(1, y)$  
2. $2x + 4y + 5z = 11$  
3. a) $x = 3 + t$, $y = 7$, $z = -2 - 2t$ is one possibility  
   b) $\frac{37}{\sqrt{56}}$
4. $x = 3 + 4t$, $y = 7 - 6t$, $z = -2 + 2t$ is one possibility
5. $y = x - \pi/3$
6. $3x - 2y = 5$
7. $42.03^\circ$, $63.47^\circ$, $74.5^\circ$, $9.01$
8. $\frac{1}{2}(13^{3/2} - 8)$
9. a) $\vec{v}(1/2) = \left< \frac{3}{\sqrt{2}}, -2\sin 1, 2\cos 1 \right>$  
   b) $\frac{2}{27}(13^{3/2} - 8)$
10. Max (-1, 2), Min (1, 4) SP (1, 2), (-1, 4)
11. (0,0) relative min, \((-1, \pm \sqrt{3})\) relative max, \((3, \pm 3)\) absolute max, \((-3, \pm 3)\) absolute min.

12. \(x = \frac{147}{3}, y = \frac{147}{3}, z = \frac{147}{3}\).

13. \(z = 38 + 4x - 27y\)

14. \(3\sqrt{13}\)

15. \(\langle -8, -4 \rangle\)

16. \(\langle -1, 1, -1 \rangle\)

17. a) 1.28
   b) \(\langle -1, 1, 0 \rangle\) is a possible answer

18. \(\frac{\partial z}{\partial x} = \frac{2x + 2z - 1 + z^2}{2x + 2zx}\)

19. 15.08 cu. in.

20. 6.72 in/min.

21. a) 0
    b) \(\frac{1}{0.4} = 2.5\)
    c) \(\frac{3}{\sqrt{2}} \approx 2.12\)

22. \(r = \sqrt[2]{\frac{V}{2\pi}}\)
    \(h = \sqrt[2]{\frac{4V}{\pi}}\)

23. \((\pm \frac{4}{3}, \frac{4}{3}, \frac{4}{3})\)

24. a) \(\int_0^2 \int_{-\frac{3}{2}}^{\frac{3}{2}} f(x, y) \, dx \, dy\)
    b) \(\int_1^2 \int_1^3 f(x, y) \, dx \, dy + \int_3^5 \int_1^3 f(x, y) \, dx \, dy + \int_5^6 \int_{-\frac{3}{2}}^{\frac{3}{2}} f(x, y) \, dx \, dy\)

25. a) \(\int_{-4}^{4} \int_{-4-z^2}^{4-z^2} 1 \, dx \, dz\)
    b) \(\int_{-4}^{4} \int_{-4-z^2}^{4-z^2} dy \, dz\)

26. 150.48

27. 34.35

28. 1/12