MAT 271

TIPS ON CHECKING FOR ABSOLUTE CONVERGENCE, CONDITIONAL CONVERGENCE OR DIVERGENCE (Sections 11.1-1.6).

We need first to distinguish between series with all positive terms and series with positive and negative terms.

SERIES WITH POSITIVE TERMS:

If the series has only positive terms than the series is either absolutely convergent or divergent.

- if you can see at a glance that the limit of the terms is not zero than the divergence theorem should be used

**EXAMPLE 1**: \( \sum_{n=1}^{\infty} \frac{2^n}{n} \) is divergent by the divergence test since (by l'Hospital) \( \lim_{n \to \infty} \frac{2^n}{n} = \lim_{n \to \infty} \frac{2^n \ln(2)}{1} = \infty \)

- if the series contains exponentials and/or factorials the ratio test can be used.

**EXAMPLE 2:** \( \sum_{n=1}^{\infty} \frac{n^2+3n-2}{2^n(3n^2+4)} \)

\[
\lim_{n \to \infty} \frac{a_{n+1}}{a_n} = \lim_{n \to \infty} \frac{((n+1)^2+3(n+1)-2)}{2^{n+1}(3(n+1)^2+4)} \cdot \frac{2^n(3n^2+4)}{n^2+3n-2} = \lim_{n \to \infty} \frac{n^2+5n+2}{2^n(3n^2+6n+7)} \cdot \frac{3n^2+4}{n^2+3n-2} = \frac{1}{4}
\]

Since the limit is less than one the series is Absolutely convergent.

- if the series contains only algebraic or rational expression of \( n \) then the integral test or the comparison test or the limit comparison test should be used. In particular, if the series contains sine or cosine functions (but still with positive terms) the comparison test should be used. If the series contains logarithmic functions the integral test or the comparison test should be used. If the inequalities are not obvious, the limit comparison test should be used. In order to find an appropriate series to compare with, you should determine the dominant terms for numerator and denominator and neglect all the remaining terms. Remember that, when using the comparison test, if you want to prove divergence you need to find a divergent series which is smaller than the given series and when proving convergence you need to find a convergent series which is larger than the given one.

**EXAMPLE 3:** \( \sum_{n=2}^{\infty} \frac{4}{n(\ln n)^2} \). Since \( \int_{2}^{\infty} \frac{4}{x(\ln x)^2} dx \) is fairly easy to evaluate we should apply the integral test. The value of the integral is finite (it equals \( 4/\ln(2) \)) thus the series is AC.

**EXAMPLE 4:** \( \sum_{n=1}^{\infty} \frac{2+\sin(n)}{n^2} \). Since we have a sine function we should try to use the comparison test. Since \(-1 \leq \sin(n) \leq 1\) we have that all the terms are positive and \( \sum_{n=1}^{\infty} \frac{2+\sin(n)}{n^2} < \sum_{n=1}^{\infty} \frac{3}{n^2} \) so the series is convergent by comparison with a convergent p-series.

**EXAMPLE 5:** \( \sum_{n=1}^{\infty} \frac{n^2+\sqrt{n}+7}{n^3+n+3n+2} \). The dominant terms are \( \frac{n^2}{n^3+n} = \frac{1}{n^{1/3}} \). We can use the limit comparison test with \( b_n = \frac{1}{n^{1/3}} \):

\[
\lim_{n \to \infty} \frac{a_n}{b_n} = \lim_{n \to \infty} \frac{n^2+\sqrt{n}+7}{n^{1/3}+3n+2n^{1/3}} = 1. \]

Since the limit is a nonzero positive number and since \( \sum_{n=1}^{\infty} \frac{1}{n^{1/3}} \) converges, we have that our series converges by the limit comparison.
SERIES WITH POSITIVE AND NEGATIVE TERMS:

If a series has positive and negative terms then it could be conditionally convergent. However you should always check for Absolute convergence first since Absolute convergence implies convergence.

- If you can see at a glance that the limit of the terms is not zero than the divergence test should be used

**EXAMPLE 6:**

\[ \sum_{n=1}^{\infty} \frac{(-1)^n 2^n}{n} \]  

From Example 1 we know that \( \lim_{n \to \infty} \frac{2^n}{n} = \infty \) thus the series is divergent by the divergence test.

- If the series contains exponentials and/or factorials the ratio test should be used. If the limit of the ratio is less than one, the series is AC. If the limit is greater than one the series is divergent and you don't need any other test.

**EXAMPLE 7:**

\[ \sum_{n=1}^{\infty} \frac{(-1)^n (n^2 + 3n - 2)}{2^n (3n^3 + 4)} \]  

From Example 2 we know that \( \lim_{n \to \infty} \frac{a_{n+1}}{a_n} = \frac{1}{4} \) thus the series is AC by the ratio test.

**EXAMPLE 8:**

\[ \sum_{n=1}^{\infty} (-1)^n \frac{(2n+1)!}{2^n} \]  

The ratio test gives:

\[ \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \to \infty} \frac{(2n+3)!}{2^{2n+2} (2n+1)!} \cdot \frac{2^n}{2^n} = \lim_{n \to \infty} \frac{(2n+3)(2n+2)}{2^2} = \infty \]  

Thus the series is divergent by the ratio test.

- If the series contains only algebraic or rational expressions in \( n \) then the integral test or the comparison test or the limit comparison test should be used to check for AC. If the series is not AC and alternating then the Alternating series test should be used to check for Conditional convergence.

**EXAMPLE 9:**

\[ \sum_{n=2}^{\infty} \frac{4(-1)^n}{n(\ln n)^2} \]  

Since \( \sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2} \) we know from EXAMPLE 3 that the series is AC by the integral test.

**EXAMPLE 10:**

\[ \sum_{n=1}^{\infty} \frac{(-1)^n}{6+n} \]  

Since \( \sum_{n=1}^{\infty} \frac{1}{6+n} \) is divergent (this can be easily checked by the integral test or the limit comparison test) we have that the series is NOT AC. To check for CC we use the Alternating series test. The conditions (i) and (ii) are satisfied so the series is Conditionally Convergent.

**EXAMPLE 11:**

\[ \sum_{n=1}^{\infty} \frac{\sin(n)}{n^3} \]  

Note that the series is not alternating but it has positive and negative terms.

Again, we look at the series with the absolute values. We have \( \sum_{n=1}^{\infty} \frac{\left| \sin(n) \right|}{n^3} \leq \sum_{n=1}^{\infty} \frac{1}{n^3} \) thus the series is AC by comparison with a convergent \( p \) series.