Def: A probability space is a triple \((S, \mathcal{F}, \mu)\) with a set \(S\), a \(\sigma\)-algebra \(\mathcal{F}\) of subsets of \(S\), and a probability distribution \(\mu\) on \(\mathcal{F}\).

Def: A random variable is a function \(X: (S, \mathcal{F}, \mu) \rightarrow \mathbb{R}\).

Examples:

1) \(S = \{1, 2, 3, 4\}\), \(\mathcal{F} = 2^S\)
   \(\mu(S_i) = \frac{1}{4}\) for \(i = 1, 2, 3, 4\)
Examples

l(a) $X(i) = i$

l(b) $X(i) = i^2$

l(c) $X(i) = 1$

l(d) $X(i) = 1 + (0.00001)i$

random, if you don't know i.

random variable which is not random

random variable which is not very random

2) $S = [0,1]$, $\mathcal{F} =$ Borel sigma algebra

$\mu =$ Lebesgue measure

2a) $X = x =$ coordinate

2b) $X = x^2$

2c) $X = \begin{cases} 0 & x < \frac{1}{2} \\ 1 & x \geq \frac{1}{2} \end{cases}$
3) \( S = \mathbb{N} \), \( \mathcal{F} = \mathcal{P}(2^S) \)
\[ \mathcal{M} = \text{Geometric Distribution} \]

3a) \( X(i) = \begin{cases} 0 & \text{if } i \text{ is even} \\ 1 & \text{otherwise} \end{cases} \)

3b) \( X(i) = i \)

3c) \( X(i) = i^2 \)

Definition: The expectation of a random variable \( X \) is (when \( X \) is discrete, i.e. \( \mathcal{M} \) is discrete)
\[
\mathbb{E}[X] = \sum_i X(i) \mathcal{P}(i)
\]

\( \mathcal{P}(i) \) probability mass function
Examples

1.a) \[ \sum_{i=1}^{4} i \frac{1}{4} = \frac{1+2+3+4}{4} = \frac{10}{4} = EX \]

1.b) \[ \sum_{i=1}^{4} i^2 \frac{1}{4} = \frac{1+4+9+16}{4} = \frac{30}{4} \]

1.c) \[ EX = 1 \]

1.d) \[ EX = 1 + 0.00001 \cdot \frac{10}{4} \]

3.a) \[ EX = \sum_{i=1}^{\infty} \begin{cases} \frac{1}{i} & \text{if } i \text{ is even} \\ \frac{1}{i} & \text{if } i \text{ is odd} \end{cases} 2^{-i} \]

\[ = \sum_{i=1}^{\infty} 2^{-(i-1)} + 0 \sum_{i=1}^{\infty} 2^{-2i} \]

\[ = \frac{2^{1/3}}{3} \]

3.b) \[ EX = \sum_{i=1}^{\infty} i 2^{-i} = \frac{1}{2} \sum_{i=1}^{\infty} i \cdot 2^{-(i-1)} \]

\[ = \frac{1}{2} \left( \sum_{i=0}^{\infty} \frac{d}{dx} x^i \right) \bigg|_{x=\frac{1}{2}} \]

\[ = \frac{1}{2} \left( \frac{d}{dx} \sum_{i=0}^{\infty} x^i \right) \bigg|_{x=\frac{1}{2}} \]
\[ = \frac{1}{2} \left( \frac{1}{1 + x} \right) \bigg|_{x=\frac{1}{2}} \]

\[ = \frac{1}{2} \left( \frac{1}{1 - \frac{1}{2}} \right) = \frac{1}{2} \cdot 4 = 2 \]

3c) \[ E(X) = \sum_{i=1}^{\infty} i^2 \cdot 2^{-i} \]

\[ = \sum_{i=1}^{\infty} i^2 \cdot x^i \bigg|_{x=\frac{1}{2}} \]

\[ = \left[ \frac{1}{4} \sum_{i=0}^{\infty} (i-1)x^{i-2} + \frac{1}{4} \sum_{i=0}^{\infty} x^{i-1} \right]_{x=\frac{1}{2}} \]

\[ = \left( \frac{1}{4} \sum_{i=0}^{\infty} \frac{d^2}{dx^2} x^i + \frac{1}{4} \sum_{i=0}^{\infty} \frac{d}{dx} x^i \right) \bigg|_{x=\frac{1}{2}} \]

\[ = \left( \frac{1}{4} \frac{d^2}{dx^2} \sum_{i=0}^{\infty} x^i + \frac{1}{4} \frac{d}{dx} \sum_{i=0}^{\infty} x^i \right) \bigg|_{x=\frac{1}{2}} \]

\[ = \frac{1}{4} \frac{d^2}{dx^2} \frac{1}{1-x} + \frac{1}{4} \frac{d}{dx} \frac{1}{1-x} \bigg|_{x=\frac{1}{2}} \]

\[ = \frac{1}{4} \frac{2}{(1-x)^3} + \frac{1}{4} \frac{1}{(1-x)^2} \bigg|_{x=\frac{1}{2}} \]

\[ = \frac{1}{4} \left( 16 + 4 \right) = 5 \]
Another example of a probability space and random variable

\[ S = \mathbb{R}, \quad \mathcal{F} = \text{Borel} \sigma\text{-algebra} \]

\[ \mu(E \times \mathbb{R}) = 2^{-x} \quad \text{if } x \text{ is a natural number} \]

\[ \mu(E \times \mathbb{R}) = 0 \quad \text{otherwise} \]

(An extension of geometric distribution to all \( \mathbb{R} \))

(I lied to you before)

**Definition:** a function \( X \) on a probability space \( (\mathbb{R}^2, \mathcal{F}, \mu) \) is measurable if for every open set \( U \) in \( \mathbb{R}^2 \)

\[ X^{-1}(U) = \{ s \in S : X(s) \in U \} \in \mathcal{F}. \]
Example

1. $S = \mathbb{N}$, $\mathcal{F} = 2^S$, $\mu = \text{geometric dist.}$

   $X(i) = i$ is measurable
   since $X^{-1}(U)$ is a set and all subsets of $S$ are in $\mathcal{F}$

2. $S = \mathbb{N}$, $\mathcal{F} = \{\emptyset, 2N, 2N-1, S\}$

   $\mu(\emptyset) = 0$, $\mu(2N) = \frac{1}{3}$, $\mu(2N-1) = \frac{2}{3}$
   $\mu(S) = 0$

   $X(i) = i$ is not measurable
   since $X^{-1}(\{\frac{1}{3}, \frac{2}{3}\}) = \{1, 2\} \neq 2N, 2N-1, S, \emptyset$

   $X(i) = \left\{ \begin{array}{ll} 0 & \text{if } i \text{ even} \\ 1 & \text{if } i \text{ odd} \end{array} \right.$

   is measurable
Definition: The distribution of a measurable random variable $X$ is the distribution on $\mathbb{R}$ which has $D_x(U) = \mu(X^{-1}U)$ for every open set $U \subseteq \mathbb{R}$.

Example: $S = (N, \mathcal{G} = 2^S)$, $\mathbb{E}i^3 = 2-i$

$X(i) = i^2$

$D_x(U) = \sum_{i: i^2 \in U} 2^{-i}$

For discrete random variable $X$

Definition $EX = \sum_{x \in \mathbb{R}} x D_x(X^{-1}x)$
The cumulative distribution of a measurable random variable $X$ is 
$$X(x) = \mu(\xi: X(\xi) \leq x)$$

2) A random variable is continuous if $X(x)$ is continuous.

3) A random variable is absolutely continuous if there is a non-negative function $ho: \mathbb{R} \rightarrow \mathbb{R}$ such that 
$$X(x) = \int_{-\infty}^{x} \rho(t) dt$$

4) A random variable is discrete if $X(x)$ is piecewise constant.

Note: as $X(1) = 1$

The expectation of a random variable is
$$\lim_{\Delta \rightarrow 0} \sum_{i = -\infty}^{\infty} i \Delta [X(i + \Delta) - X(i \Delta)]$$
Def 1: Let \( A \in \mathcal{F} \) satisfy \( \mu(A) > 0 \). The conditional probability distribution of \( \mu \) on \( A \) is defined as follows:

for \( B \in \mathcal{F} \)

\[
\mu(B|A) = \frac{\mu(B \cap A)}{\mu(A)}
\]

Note \( \mu(A^c|A) = 0 \)

\( \mu(A|A) = 1 \)