Step Functions, Shifting and Laplace Transforms

The basic step function (called the Heaviside Function) is

$$u_c(t) = \begin{cases} 0, & t < c \\ 1, & t \geq c \end{cases}$$

It is “off” (0) when $t < c$, the “on” (1) when $t \geq c$. Don’t let the notation confuse you. The function $u_c(t)$ is either 0 and 1, nothing more.

If $f(t)$ is a function, then we can shift it so that it “starts” at $t = c$. This results in the function

$$y = \begin{cases} 0, & t < c \\ f(t-c), & t \geq c \end{cases}$$

Using the step notation, this same function (now called $g(t)$) can be written

$$g(t) = u_c(t)f(t-c).$$

The Laplace Transform is

$$L\{g(t)\} = L\{u_c(t)f(t-c)\} = e^{-cs}L\{f(t)\}.$$  

Be sure the shift is already accounted for beforehand, then take the transform of the function as normally done.

**Example:** Find the Laplace transform of $y = u_2(t)(t-2)^2$.

**Solution:** Here, $y = 0$ for $t < 2$, then $u_2(t) = 1$ for $y \geq 2$. Thus, it turns “on” the function $(t-2)^2$, which is the graph of $f(t) = t^2$ shifted 2 units to the right. This is in the correct form to use the Laplace transform. The Laplace Transform of $t^2$ is $L\{t^2\} = \frac{2}{s^3}$. Therefore, the Laplace transform of $g(t) = u_2(t)(t-2)^2$ is

$$L\{g(t)\} = L\{u_2(t)(t-2)^2\} = e^{-2s}L\{t^2\} = e^{-2s} \left( \frac{2}{s^3} \right) = \frac{2e^{-2s}}{s^3}.$$  

**Example:** Find the Laplace Transform of $y = u_3(t)t^2$.

**Solution:** Note that this is not the function $t^2$ shifted right 3 units. This is actually the function $t^2$ still centered at $t = 0$, staying “off” until $t = 3$, then it turns on. In other words, $t^2$ has not been shifted right 3 units like the form requires.

We work in the shift as follows:

$$g(t) = u_3(t)(t-3+3)^2$$

We expand $(t-3+3)^2$ by grouping $t-3$ as one term, 3 as the other term:

$$(t-3+3)^2 = (t-3)^2 + 6(t-3) + 9.$$
So the function becomes \( g(t) = u_3(t)t^2 = u_3(t)(t - 3 + 3)^2 = u_3(t)[(t - 3)^2 + 6(t - 3) + 9] \).

Now we can use the Laplace Transform formulas:

\[
L\{u_3(t)[(t - 3)^2 + 6(t - 3) + 9]\} = e^{-3s}L\{t^2\} + 6L\{t\} + 9 = e^{-3s}\left(\frac{2}{s^3} + \frac{6}{s^2} + \frac{9}{s}\right).
\]

Example: Find \( L\{u_{\pi/3}(t) \sin \left( t - \frac{\pi}{3}\right)\} \).

Solution: The graph of \( y = u_{\pi/3}(t) \sin \left( t - \frac{\pi}{3}\right) \) is the sine function shifted to the right \( \frac{\pi}{3} \) units, as shown below:
The shift is accounted for, so

\[
L \left\{ u_{\pi/3}(t) \sin \left( t - \frac{\pi}{3} \right) \right\} = e^{-(\pi/3)s} L\{\sin t\}
\]
\[
= e^{-(\pi/3)s} \left( \frac{1}{s^2 + 1} \right)
\]
\[
= \frac{e^{-(\pi/3)s}}{s^2 + 1}.
\]

**Example:** Find \( L\{u_{\pi/3}(t) \sin t\} \).

**Solution:** The graph of \( y = u_{\pi/3}(t) \sin t \) is shown below. Note that it is the sine function, but not shifted. It simply is “off” until \( t = \frac{\pi}{3} \), at which time it turns “on” and follows the graph of \( \sin t \) as normal.

![Graph of sine function with shift](image)

The shift \( t - \frac{\pi}{3} \) is not present in the \( \sin t \) as is needed to perform this transform. We put it in as follows:

\[
\sin t = \sin \left( t - \frac{\pi}{3} + \frac{\pi}{3} \right).
\]
We then use the sine-sum identity, \( \sin(a + b) = \sin a \cos b + \cos a \sin b \):

\[
\sin \left[ \left( t - \frac{\pi}{3} \right) + \frac{\pi}{3} \right] = \sin \left( t - \frac{\pi}{3} \right) \cos \frac{\pi}{3} + \cos \left( t - \frac{\pi}{3} \right) \sin \frac{\pi}{3}.
\]

Recall from trigonometry that \( \cos \frac{\pi}{3} = \frac{1}{2} \) and that \( \sin \frac{\pi}{3} = \frac{\sqrt{3}}{2} \). Therefore,

\[
\sin \left[ \left( t - \frac{\pi}{3} \right) + \frac{\pi}{3} \right] = \sin \left( t - \frac{\pi}{3} \right) \cos \frac{\pi}{3} + \cos \left( t - \frac{\pi}{3} \right) \sin \frac{\pi}{3} = \frac{1}{2} \sin \left( t - \frac{\pi}{3} \right) + \frac{\sqrt{3}}{2} \cos \left( t - \frac{\pi}{3} \right).
\]

Therefore, to find \( L\{u_{\pi/3}(t) \sin t\} \), we find

\[
L\left\{ u_{\pi/3}(t) \left[ \frac{1}{2} \sin \left( t - \frac{\pi}{3} \right) + \frac{\sqrt{3}}{2} \cos \left( t - \frac{\pi}{3} \right) \right] \right\} = L\left\{ u_{\pi/3}(t) \sin t \right\} = L\left\{ \sin \left( t - \frac{\pi}{3} \right) \right\}.
\]

Recall that \( L\{\sin t\} = \frac{1}{s^2 + 1} \) and that \( L\{\cos t\} = \frac{s}{s^2 + 1} \). Since the shifts are now accounted for, we have:

\[
L\left\{ u_{\pi/3}(t) \sin t \right\} = L\left\{ u_{\pi/3}(t) \sin t \right\} = L\left\{ \frac{1}{2} \sin \left( t - \frac{\pi}{3} \right) + \frac{\sqrt{3}}{2} \cos \left( t - \frac{\pi}{3} \right) \right\} = e^{-\left(\pi/3\right)s} \left[ \frac{1}{2} L\{\sin t\} + \frac{\sqrt{3}}{2} L\{\cos t\} \right] = e^{-\left(\pi/3\right)s} \left[ \frac{1}{2} \left( \frac{1}{s^2 + 1} \right) + \frac{\sqrt{3}}{2} \left( \frac{s}{s^2 + 1} \right) \right] = \frac{1}{2} e^{-\left(\pi/3\right)s} \left( \frac{1 + \sqrt{3}s}{s^2 + 1} \right).
\]