Autonomous Differential Equations
Population Dynamics
Common Types: ① Uninhibited ② Inhibited ③ Logistic

Uninhibited case: population can grow “forever.”

The rate of change of a population is directly proportional to the population.
Suppose the initial population is 250, find $P(t)$, the population at time $t$ (years).

\[
\frac{dP}{dt} = kP \Rightarrow \frac{dP}{P} = k \, dt \Rightarrow \int \frac{dP}{P} = \int k \, dt = \ln P = (kt + c)
\]

\[
\Rightarrow P = Ce^{kt}
\]

To find $C$, use the point $P(0)=250 \Rightarrow 250 = Ce^{k(0)} \Rightarrow C=250
\]

\[
\Rightarrow P = 250e^{kt}
\]

Suppose the population is 500 after 3 yrs. (Now we can find $k$).

\[
500 = 250e^{k(3)}
\]

\[
\Rightarrow 2 = e^{3k}
\]

\[
\Rightarrow \ln 2 = 3k \Rightarrow k = \frac{\ln 2}{3} \approx 0.231
\]

\[
P(t) = 250e^{0.231t}
\]

Uninhibited growth (nothing to stop it).
Inhibited growth model.
Assume that \( L \) is the limiting population. In this case, the rate of change of the population is directly proportional to the available "room for growth."

\[
\frac{dp}{dt} = k(L-p) \quad (L \text{ is a constant})
\]

\[
\frac{dp}{L-p} = k \, dt \Rightarrow \int \frac{dp}{L-p} = \int k \, dt
\]

\[
-\ln(L-p) = kt + C
\]

\[
\ln(L-p) = (C - kt) \quad (C \text{ absorbs the negative, } k \text{ does not})
\]

\[
L-p = Ce^{-kt}
\]

\[
-p = Ce^{-kt} - L
\]

\[
p(t) = L + Ce^{-kt}
\]

Suppose at time \( t=0 \), the population was 1000, and the limiting population is 10,000. After 3 yrs, the population was 2500.

\[
L = 10,000:
\]

\[ p(0) = 1000 \]

\[
1000 = 10,000 + Ce^{-k(0)} \Rightarrow C = -9000
\]

\[
p(t) = 10000 - 9000e^{-kt}
\]

\[
2500 = 10,000 - 9000e^{-k(3)}
\]

\[
k = \frac{\ln\left(\frac{7500}{7000}\right)}{-3} \approx 0.0608
\]

\[
p(t) = 10,000 - 9000e^{-0.0608t}
\]
Logistics Growth Model

The rate of change of a population is directly proportional to both the population and the available room for growth. Assume \( L \) = limiting population.

\[
\frac{dP}{dt} = KP(L-P) \Rightarrow \frac{dP}{P(L-P)} = k \, dt
\]

\[
\int \frac{dP}{P(L-P)} = \int k \, dt
\]

**PARTIAL FRACTIONS!**

\[
\frac{1}{P(L-P)} = \frac{a}{P} + \frac{b}{L-P}
\]

\[
\Rightarrow \frac{1}{P(L-P)} = \frac{a(L-P) + bP}{P(L-P)} = \frac{aL-aP+bP}{P(L-P)}
\]

\[
\Rightarrow \frac{1}{L} = a + \frac{b}{L}
\]

\[
p(L) + aL = \frac{1}{L}
\]

\[
a = \frac{1}{L}
\]

So therefore,

\[
\int \frac{dP}{P(L-P)} = \int \left( \frac{1}{L} \right) \frac{dP}{P} + \int \frac{1}{L-P} \, dp
\]

\[
= \frac{1}{L} \left( \ln P - \ln (L-P) \right) = \frac{1}{L} \ln \left( \frac{P}{L-P} \right)
\]

Remember, we had

\[
\int \frac{dP}{P(L-P)} = \int k \, dt
\]

\[
\Rightarrow \frac{1}{L} \ln \left( \frac{P}{L-P} \right) = kt + C
\]

\[
\Rightarrow \ln \left( \frac{P}{L-P} \right) = (Lkkt + C)
\]

\[
\Rightarrow \frac{P}{L-P} = Ce^{Lkkt}
\]

\[
\Rightarrow P = CE^{Lkkt}(L-P)
\]

\[
\Rightarrow P = LCe^{Lkkt} - PCe^{Lkkt}
\]

\[
\Rightarrow P + PCe^{Lkkt} = LCe^{Lkkt}
\]

\[
P(1 + Ce^{Lkkt}) = LCe^{Lkkt}
\]

\[
P(t) = \frac{LCe^{Lkkt}}{1 + Ce^{Lkkt}}
\]

\[
\Rightarrow \frac{LP(t)}{L} = Ce^{-Lkkt}
\]

\[
\text{Logistics Function}
\]

\[
\Rightarrow \text{Logistic Growth}
\]
Logistics Model: \( P(t) = \frac{L}{1 + Ce^{-kt}} \)

Suppose a cruise ship holds 5000 people. At time \( t=0 \), 10 people get sick. 48 hrs later, 200 people are sick.

So \( L = 5000 \)...

Initial population is \( P(0) = 10 \)

\[
10 = \frac{5000}{1 + Ce^{-2k(0)}} \Rightarrow 10 = \frac{5000}{1 + C} \Rightarrow C = 499
\]

\[
\therefore P(t) = \frac{5000}{1 + 499e^{-5000kt}}.
\]

Now find \( k \):

\[
200 = \frac{5000}{1 + 499e^{-5000k(48)}} \Rightarrow \frac{499e^{-240000k}}{499} = 2.4 \Rightarrow e^{-240000k} = \frac{24}{499} \Rightarrow -240000k = \ln\left(\frac{24}{499}\right) \Rightarrow k = \frac{\ln\left(\frac{24}{499}\right)}{-240000} \approx 0.00001264...
\]

So \( P(t) = \frac{5000}{1 + 499e^{-5000\left(0.00001264\right)t}} \)

\[
(P(t) = \frac{5000}{1 + 499e^{-0.0632t}})
\]