Proof of the Magnitude of a Cross Product, $|\mathbf{u}\times\mathbf{v}| = |\mathbf{u}||\mathbf{v}|\sin\theta$

Let $\mathbf{u} = \langle u_1, u_2, u_3\rangle$ and $\mathbf{v} = \langle v_1, v_2, v_3\rangle$ be two vectors in R^3, and let θ be the angle that the two vectors form when their feet are placed together. The cross product of **u** and **v** is

$$\mathbf{u}\times\mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix}\mathbf{i} - \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix}\mathbf{j} + \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix}\mathbf{k} = \langle u_2v_3 - u_3v_2,\ u_3v_1 - u_1v_3,\ u_1v_2 - u_2v_1\rangle.$$

Therefore, the magnitude-squared of the cross product is

$$\begin{aligned} |\mathbf{u}\times\mathbf{v}|^2 &= (\mathbf{u}\times\mathbf{v})\cdot(\mathbf{u}\times\mathbf{v}) \\ &= (u_2v_3 - u_3v_2)^2 + (u_3v_1 - u_1v_3)^2 + (u_1v_2 - u_2v_1)^2 \\ &= u_2^2v_3^2 - 2u_2u_3v_2v_3 + u_3^2v_2^2 + u_3^2v_1^2 - 2u_1u_3v_1v_3 + u_1^2v_3^2 + u_1^2v_2^2 - 2u_1u_2v_1v_2 + u_2^2v_1^2. \quad \textbf{(A)} \end{aligned}$$

Meanwhile,

$$\begin{aligned} (|\mathbf{u}||\mathbf{v}|)^2 &= \left(\sqrt{u_1^2 + u_2^2 + u_3^2}\sqrt{v_1^2 + v_2^2 + v_3^2}\right)^2 \\ &= (u_1^2 + u_2^2 + u_3^2)(v_1^2 + v_2^2 + v_3^2) \\ &= u_1^2v_1^2 + u_1^2v_2^2 + u_1^2v_3^2 + u_2^2v_1^2 + u_2^2v_2^2 + u_2^2v_3^2 + u_3^2v_1^2 + u_3^2v_2^2 + u_3^2v_3^2. \quad \textbf{(B)} \end{aligned}$$

Also, notice that

$$\begin{aligned} (\mathbf{u}\cdot\mathbf{v})^2 &= (u_1v_1 + u_2v_2 + u_3v_3)^2 \\ &= u_1^2v_1^2 + u_2^2v_2^2 + u_3^2v_3^2 + 2u_1u_2v_1v_2 + 2u_1u_3v_1v_3 + 2u_2u_3v_2v_3. \quad \textbf{(C)} \end{aligned}$$

If you look very closely, you'll see that line **(C)** subtracted from **(B)** gives **(A)**. This is called Lagrange's Identity. Thus, we have

$$\begin{aligned} |\mathbf{u}\times\mathbf{v}|^2 &= (|\mathbf{u}||\mathbf{v}|)^2 - (\mathbf{u}\cdot\mathbf{v})^2 \\ &= (|\mathbf{u}||\mathbf{v}|)^2 - (|\mathbf{u}||\mathbf{v}|\cos\theta)^2 \quad \text{Using the definition of the dot product} \\ &= (|\mathbf{u}||\mathbf{v}|)^2 - (|\mathbf{u}||\mathbf{v}|)^2\cos^2\theta \\ &= (|\mathbf{u}||\mathbf{v}|)^2(1 - \cos^2\theta) \\ &= (|\mathbf{u}||\mathbf{v}|)^2\sin^2\theta. \end{aligned}$$

Taking square roots, we have $|\mathbf{u}\times\mathbf{v}| = |\mathbf{u}||\mathbf{v}|\sin\theta$. Since two vectors never open wider than 180 degrees (π radians), $\sin\theta$ will be positive.

A nice corollary is the relationship

$$(|\mathbf{u}||\mathbf{v}|)^2 = |\mathbf{u}\times\mathbf{v}|^2 + (\mathbf{u}\cdot\mathbf{v})^2.$$

I'm not sure what the significance is, but it looks pretty.

If you see an error, please email me at surgent@asu.edu. (Prepared by Scott Surgent, 12-20-2012, edited 1-24-14 and again 11-16-15)