Minimum Trigonometric Knowledge Required For Calculus

Trigonometry can seem like hundreds of formulas and identities, but in reality you don’t need to memorize every single formula. What follows is a reasonable “base-line” knowledge level that should be adequate for calculus. The more you use it, the better it stays with you (and makes more sense).

I. Definitions of the Trigonometric Functions.

A circle is drawn with radius 1 and center at the origin. This is called the unit circle. A ray is drawn from the origin outward. It intersects the circle at point \( P \) with angle \( \theta \). By definition, the \( x \)-coordinate of this point is the cosine of \( \theta \) (abbreviated \( \cos \theta \)), and the \( y \)-coordinate is the sine of \( \theta \) (abbreviated \( \sin \theta \)). The slope of the ray is the tangent of \( \theta \) (abbreviated \( \tan \theta \)).

There are three other trigonometric functions: the secant of \( \theta \) (abbreviated \( \sec \theta \)) is the reciprocal of \( \cos \theta \); the cotangent of \( \theta \) (abbreviated \( \cot \theta \)) is the reciprocal of \( \tan \theta \); and the cosecant of \( \theta \) (abbreviated \( \csc \theta \)) is the reciprocal of \( \sin \theta \).

II. Quadrants and Signs of the Trigonometric Functions.

The \( xy \)-plane is split into four quadrants, numbered I, II, III and IV as shown in the figure below.

i) When the ray is in the first quadrant, the \( x \) and \( y \) coordinates of point \( P \) are positive, thus \( \cos \theta \) and \( \sin \theta \) are positive. The ray also has a positive slope hence \( \tan \theta \) is positive. In fact, all six trigonometric functions are positive in Q-I.

ii) When the ray is in the second quadrant, the \( x \) coordinate of point \( P \) is negative and the \( y \) coordinate is positive. Thus, \( \cos \theta \) is negative and \( \sin \theta \) is positive. The slope is negative, hence \( \tan \theta \) is negative.

iii) When the ray is in the third quadrant, the \( x \) and \( y \) coordinates of point \( P \) are both negative, so both \( \cos \theta \) and \( \sin \theta \) are negative. The ray has a positive slope so \( \tan \theta \) is positive.

iv) When the ray is in the fourth quadrant, the \( x \) coordinate of point \( P \) is positive and the \( y \) coordinate negative, so \( \cos \theta \) is positive and \( \sin \theta \) is negative. The slope is negative so \( \tan \theta \) is negative.

A mnemonic is ASTC “All Students Take Calculus”. A = all trig function are positive in Q-I; S = sine is positive in Q-II; T = tan is positive in Q-III; and C = cos is positive in Q-IV.
III. Radian and Degree Measure.

The circle is divided into 360 degrees, which is a nice number but has no physically meaningful interpretation. An alternative measurement called radian measure uses the circumference of the unit circle so that the angle of the ray that “subtends” the circle is equal to the length of the arc it creates.

The circumference formula for a circle is $C = 2\pi r$. For the unit circle, the radius is 1, so the circumference is $C = 2\pi$. Since this corresponds to a complete circle (360 degrees), we declare that

$$360 \text{ degrees} = 2\pi \text{ radians}$$

These figures are proportional, so that half a circle—180 degrees—is $\pi$ radians, and quarter circle—90 degrees—is $\frac{\pi}{2}$ radians, and so forth. The usual conversion from degrees to radians is to multiply the degree figure by $\frac{\pi}{180}$. Below are some common degree-radian conversions you should know:

<table>
<thead>
<tr>
<th>Degrees</th>
<th>Radians</th>
</tr>
</thead>
<tbody>
<tr>
<td>360</td>
<td>$2\pi$</td>
</tr>
<tr>
<td>270</td>
<td>$\frac{3}{2}\pi$</td>
</tr>
<tr>
<td>180</td>
<td>$\pi$</td>
</tr>
<tr>
<td>90</td>
<td>$\frac{\pi}{2}$</td>
</tr>
<tr>
<td>60</td>
<td>$\frac{\pi}{3}$</td>
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<tr>
<td>45</td>
<td>$\frac{\pi}{4}$</td>
</tr>
<tr>
<td>30</td>
<td>$\frac{\pi}{6}$</td>
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<tr>
<td>0</td>
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IV. Exact First-Quadrant Measures.

You should know the exact measurements for the common first-quadrant angles for $\cos \theta$ and $\sin \theta$. Knowing them you can then construct the exact values for $\tan \theta$. These measurements are based on the familiar 30-60-90 and 45-45-90 triangles and the Pythagorean Theorem. Here’s a neat way to set them up:

Set up three columns, one for the angles going from 0 up to 90 degrees (0 to $\frac{\pi}{2}$ radians). The second column is for $\cos \theta$ and the third column is for $\sin \theta$. Fill in blank roots over 2 for all entries. Then write in the numbers 4-3-2-1-0 for $\cos$, and 0-1-2-3-4 for $\sin$, and reduce what you can!

<table>
<thead>
<tr>
<th>$\theta$</th>
<th>$\cos \theta$</th>
<th>$\sin \theta$</th>
<th>$\theta$</th>
<th>$\cos \theta$</th>
<th>$\sin \theta$</th>
<th>$\theta$</th>
<th>$\cos \theta$</th>
<th>$\sin \theta$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$\frac{\sqrt{2}}{2}$</td>
<td>$\frac{\sqrt{2}}{2}$</td>
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<td>$\frac{\sqrt{2}}{2}$</td>
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<tr>
<td>30 ($\frac{\pi}{6}$)</td>
<td>$\frac{\sqrt{3}}{2}$</td>
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<tr>
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<td>0</td>
<td>1</td>
<td>90 ($\frac{\pi}{2}$)</td>
<td>0</td>
<td>1</td>
</tr>
</tbody>
</table>

Converting angles in other quadrants is just a matter of using symmetry and remembering the sign values that are appropriate depending on the quadrant. For example, given $\cos \frac{4\pi}{3}$, we would note that the angle $\frac{4\pi}{3}$ is in
Q-III, and that cos is negative in Q-III, and that angle $\frac{4\pi}{3}$ is symmetric to $\frac{\pi}{3}$ in quadrant. Looking up $\cos\frac{\pi}{3}$, we can then determine that $\cos\frac{4\pi}{3} = -\frac{1}{2}$.

V. Basic Identities

There are many useful trigonometric identities we use. The following should be memorized as best you can:

*Pythagorean Identities:*

\[
\sin^2 \theta + \cos^2 \theta = 1. \quad \text{Corollaries: } \sin^2 \theta = 1 - \cos^2 \theta, \quad \cos^2 \theta = 1 - \sin^2 \theta.
\]

Dividing by $\sin^2 \theta$ or $\cos^2 \theta$ gives more identities: $\tan^2 \theta + 1 = \sec^2 \theta$ and its corollary $\tan^2 \theta = \sec^2 \theta - 1$, and also $1 + \cot^2 \theta = \csc^2 \theta$ and its corollary $\cot^2 \theta = \csc^2 \theta - 1$.

*Double angle identities:*

\[
\sin(2\theta) = 2\sin \theta \cos \theta \\
\cos(2\theta) = \cos^2 \theta - \sin^2 \theta = 1 - 2\sin^2 \theta = 2\cos^2 \theta - 1
\]

*Half angle identities:*

\[
\sin^2 \theta = \frac{1 - \cos(2\theta)}{2}; \quad \cos^2 \theta = \frac{1 + \cos(2\theta)}{2}
\]

*Shift Identities:*

\[
\sin(\theta + \frac{\pi}{2}) = \cos \theta; \quad \cos(\theta + \frac{\pi}{2}) = -\sin \theta
\]

*Symmetries:*

\[
\sin(-\theta) = -\sin(\theta) \quad \text{(sine and tangent are both odd functions, symmetric with the origin)} \\
\tan(-\theta) = -\tan(\theta) \\
\cos(-\theta) = \cos(\theta) \quad \text{(cosine is an even function, symmetric with the y-axis)}
\]

We use many of these forms in integration techniques!