Articulation: Each problem on the test was covered in my typed notes (TN), in WebWork (WW) or in the practice problems (PP).

1. Given two points $A = (0,1,5)$ and $B = (2,-6,1)$.

   a) Find the vector $\mathbf{v}$ whose foot is $A$ and head (arrow) at $B$.

   * Subtract the components of $A$ from $B$.
   * Version A: $\mathbf{v} = (2-0, -6-1, 1-5) = (2, -7, -4)$
   * Version B: $\mathbf{v} = (0 - 5, 3 - 1, 2 - (-1)) = (-5, 2, 3)$
   * Articulation: TN: Ex 6.3: WW: 10.2.5, 10.5.3, 10.5.4; PP: #25

   b) Find a vector $\mathbf{w}$ that is parallel to $\mathbf{v}$ (in part (a)) but points the other way and has length 5. No decimal answers accepted.

   * Get a unit vector first, then multiply by $-5$:
   * Version A: $\mathbf{w} = -\frac{5}{\sqrt{69}}(2, -7, -4) = \left(-\frac{10}{\sqrt{69}}, \frac{35}{\sqrt{69}}, \frac{20}{\sqrt{69}}\right)$
   * Version B: $\mathbf{w} = -\frac{5}{\sqrt{38}}(-5, 2, 3) = \left(-\frac{25}{\sqrt{38}}, -\frac{10}{\sqrt{38}}, -\frac{15}{\sqrt{38}}\right)$
   * Articulation: TN: Ex 6.9; WW: 10.2.2, 10.2.3, 10.2.9, 10.2.17; PP: #13

2. In $\mathbb{R}^3$, let $A = (2, -1, 3)$ be a point and $P$ be the plane $5x + 3y - 2z = 9$.

   a) Find the equation of the line (in parametric form, with parameter variable $t$) of the line passing through $A$, orthogonal to $P$.

   * Need a point, which is $(2, -1, 3)$ and a normal vector, which is $\mathbf{n} = (5, 3, -2)$.
   * Thus, $(x(t), y(t), z(t)) = (2, -1, 3) + t(5, 3, -2)$, or $x(t) = 2 + 5t, y(t) = -1 + 3t, z(t) = 3 - 2t$.
   * Articulation: TN: 13.1; WW: 10.5.7; PP: #16, 24, 25

   b) Where does the line found in part (a) intersect plane $P$?

   * Substitute your expressions for $x$, $y$ and $z$ into the plane and solve for $t$:
   * $5(2 + 5t) + 3(-1 + 3t) - 2(3 - 2t) = 9$, which gives $10 + 25t - 3 + 9t - 6 + 4t = 9$.
   * Simplified, you get $38t + 1 = 9$, so that $38t = 8$, or $t = \frac{8}{38} = \frac{4}{19}$.
   * Now, find the point by evaluating the line at this $t$-value:
   * $x \left(\frac{4}{19}\right) = 2 + 5 \left(\frac{4}{19}\right) = \frac{58}{19} \approx 3.053$
   * $y \left(\frac{4}{19}\right) = -1 + 3 \left(\frac{4}{19}\right) = -\frac{7}{19} \approx -0.368$
   * $z \left(\frac{4}{19}\right) = 3 - 2 \left(\frac{4}{19}\right) = \frac{49}{19} \approx 2.579$
   * Articulation: TN: Ex 13.4; WW: 10.5.15
3. Let $\mathbf{u} = \langle 1, 4, 2 \rangle$ and $\mathbf{v} = \langle 3, -1, 6 \rangle$ be two vectors.

   a) Find the angle, in degrees, between $\mathbf{u}$ and $\mathbf{v}$.

   \[ \theta = \cos^{-1}\left( \frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{u}||\mathbf{v}|} \right) = \cos^{-1}\left( \frac{11}{\sqrt{21}\sqrt{46}} \right) \approx 69.27^\circ. \]

   Articulation: TN: Ex 7.1; WW: 10.3.6, 10.3.8; PP: #11

   b) Find the area of the triangle where $\mathbf{u}$ and $\mathbf{v}$ are two of its legs. No decimal answers will be permitted.

   Need the cross product: $\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 4 & 2 \\ 3 & -1 & 6 \end{vmatrix} = \langle 26, 0, -13 \rangle$.

   Version A, triangle: Area = $\frac{1}{2} |\mathbf{u} \times \mathbf{v}| = \frac{1}{2} \sqrt{845}$.

   Version B: parallelogram: Area = $|\mathbf{u} \times \mathbf{v}| = \sqrt{845}$.

   Articulation: TN: Ex 9.3 and 9.4; WW: 10.4.6, 10.4.7; PP: #18, 19

   Note: The only difference between the versions was that $\mathbf{u}$ and $\mathbf{v}$ were switched. It makes no difference in the results of parts (a) and (b).

   c) Suppose $\mathbf{u}$ is the hypotenuse of a right triangle and the adjacent leg of this right triangle is a vector parallel to $\mathbf{v}$. Find $\mathbf{v}$.

   Find the projection of $\mathbf{u}$ onto $\mathbf{v}$.

   Version A: $\text{proj}_\mathbf{v} \mathbf{u} = \left( \frac{\mathbf{u} \cdot \mathbf{v}}{\mathbf{v} \cdot \mathbf{v}} \right) \mathbf{v} = \frac{11}{46} \mathbf{v} = \left\langle \frac{33}{46}, \frac{-11}{46}, \frac{66}{46} \right\rangle$.

   Version B: $\text{proj}_\mathbf{v} \mathbf{u} = \left( \frac{\mathbf{u} \cdot \mathbf{v}}{\mathbf{v} \cdot \mathbf{v}} \right) \mathbf{v} = \frac{11}{21} \mathbf{v} = \left\langle \frac{11}{21}, \frac{44}{21}, \frac{22}{21} \right\rangle$.

   Articulation: TN: Ex 8.1; WW: 10.3.7, 10.3.9, 10.3.14; PP: #14, 15, 35

4. Find the length of the curve traced out by $\mathbf{r}(t) = \langle \sqrt{t}, 2 \cos t, 2 \sin t \rangle$ for $0 \leq t \leq \pi$. Decimal answers will be accepted. Show your integral for full credit.

   Version A: Find $\mathbf{r}'(t) = \left( \frac{1}{2\sqrt{t}}, -2 \sin t, 2 \cos t \right)$.

   Now find speed: $|\mathbf{r}'(t)| = \sqrt{\left( \frac{1}{2\sqrt{t}} \right)^2 + (-2 \sin t)^2 + (2 \cos t)^2} = \sqrt{\frac{1}{4t} + 4}$.

   Integrate from 0 to $\pi$: $\int_0^\pi \sqrt{\frac{1}{4t} + 4} \, dt \approx 6.677$ units.

   Version B: $\mathbf{r}'(t) = \left( \frac{1}{2\sqrt{t}}, -3 \sin t, 3 \cos t \right)$, $|\mathbf{r}'(t)| = \sqrt{\frac{1}{4t} + 9}, \int_0^{2\pi} \sqrt{\frac{1}{4t} + 9} \, dt \approx 19.133$ units.

   Articulation: TN: Ex 19.3; WW: 10.8.1; PP: #31

   Note: you were allowed to use your calculator to evaluate. There was no restriction against doing so in the directions. This integrand will not antidifferentiate using “normal” techniques.

   Note: If you use your calculator, you get a situation where 0 is not in the domain of the derivative, but this integral can still be evaluated as a limit (i.e. use 0.001 as a lower bound).
5. An object is attached to a string of radius 4 m and revolved in a circular motion, completing a full revolution every 12 seconds. Assume that the object starts in motion at time $t = 0$ at the point $(4,0)$ and moves in a counterclockwise motion.

a) Find a vector-valued function $\mathbf{r}(t)$ that describes the position of the object at time $t$.

- **Version A**: $\mathbf{r}(t) = \langle 4 \cos \left( \frac{2\pi}{12} t \right), 4 \sin \left( \frac{2\pi}{12} t \right) \rangle = \langle 4 \cos \left( \frac{\pi}{6} t \right), 4 \sin \left( \frac{\pi}{6} t \right) \rangle$
- **Version B**: $\mathbf{r}(t) = \langle 6 \cos \left( \frac{2\pi}{4} t \right), 6 \sin \left( \frac{2\pi}{4} t \right) \rangle = \langle 6 \cos \left( \frac{\pi}{2} t \right), 6 \sin \left( \frac{\pi}{2} t \right) \rangle$
- **Articulation**: TN: Ex. 17.6; WW: 10.9.8; PP: #38

b) Find the speed of the object.

- Need the derivative first
- **Version A**: $\mathbf{r}'(t) = \langle -4 \left( \frac{\pi}{6} \right) \sin \left( \frac{\pi}{6} t \right), 4 \left( \frac{\pi}{6} \right) \cos \left( \frac{\pi}{6} t \right) \rangle = \langle -\frac{2\pi}{3} \sin \left( \frac{\pi}{6} t \right), \frac{2\pi}{3} \cos \left( \frac{\pi}{6} t \right) \rangle$
- **Version A**: $|\mathbf{r}'(t)| = \sqrt{\left( -\frac{2\pi}{3} \sin \left( \frac{\pi}{6} t \right) \right)^2 + \left( \frac{2\pi}{3} \cos \left( \frac{\pi}{6} t \right) \right)^2} = \sqrt{\frac{4\pi^2}{9} \left( \cos^2 \left( \frac{\pi}{6} t \right) + \sin^2 \left( \frac{\pi}{6} t \right) \right)} = \frac{2\pi}{3}$
- **Version B**: $\mathbf{r}'(t) = \langle -6 \left( \frac{\pi}{2} \right) \sin \left( \frac{\pi}{2} t \right), 6 \left( \frac{\pi}{2} \right) \cos \left( \frac{\pi}{2} t \right) \rangle = \langle -3\pi \sin \left( \frac{\pi}{2} t \right), 3\pi \cos \left( \frac{\pi}{2} t \right) \rangle$
- **Version B**: $|\mathbf{r}'(t)| = 3\pi$
- **Articulation**: TN: Ex 17.6; WW: 10.7.9, 10.9.1

c) Find $\mathbf{T}(t)$.

- Use the value of $|\mathbf{r}'(t)|$ found in part (b).
- **Version A**: $\mathbf{T}(t) = \frac{\mathbf{r}(t)}{|\mathbf{r}'(t)|} = \frac{\langle -\frac{2\pi}{3} \sin \left( \frac{\pi}{6} t \right), \frac{2\pi}{3} \cos \left( \frac{\pi}{6} t \right) \rangle}{\frac{2\pi}{3}} = \langle -\sin \left( \frac{\pi}{6} t \right), \cos \left( \frac{\pi}{6} t \right) \rangle$
- **Version B**: $\frac{\langle -3\pi \sin \left( \frac{\pi}{2} t \right), 3\pi \cos \left( \frac{\pi}{2} t \right) \rangle}{3\pi} = \langle -\sin \left( \frac{\pi}{2} t \right), \cos \left( \frac{\pi}{2} t \right) \rangle$
- **Articulation**: TN: Ex. 20.3; WW: 10.8.4a
6. Short answer (5 points each)

Version A:

a) True or False: if $\mathbf{u} \times \mathbf{v} = 0$, then $\mathbf{u}$ and $\mathbf{v}$ are orthogonal.

b) Let $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ be three vectors. Circle the expression below that is not defined.

$$(\mathbf{u} \cdot \mathbf{v})\mathbf{w} \quad (\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w} \quad (\mathbf{u} \cdot \mathbf{v}) \times \mathbf{w}$$

c) What is the domain of $\mathbf{r}(t) = \langle t^2 + 5t, \sqrt{t - 4} \rangle$? $[4, \infty)$.

d) What kind of surface is $x^2 + y^2 + 2z^2 - 3x + 4y - 5z = 9$? Circle one.

Sphere Ellipsoid Hyperboloid of One Sheet Hyperboloid of Two Sheets Cylinder

Version B:

a) True or False: if $\mathbf{u} \times \mathbf{v} = 0$, then $\mathbf{u}$ and $\mathbf{v}$ are parallel.

b) Let $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ be three vectors. Circle the expression below that is not defined.

$$(\mathbf{u} \cdot \mathbf{v})\mathbf{w} \quad (\mathbf{u} \times \mathbf{v}) \times \mathbf{w} \quad (\mathbf{u} \cdot \mathbf{v}) \cdot \mathbf{w}$$

c) What is the domain of $\mathbf{r}(t) = \langle t^2 + 5t, \frac{3}{t} \rangle$? $(-\infty, 0) \cup (0, \infty)$

d) What kind of surface is $x^2 - y^2 + 2z^2 - 3x + 4y - 5z = 9$? Circle one.

Sphere Ellipsoid Hyperboloid of One Sheet Hyperboloid of Two Sheets Cylinder

Articulation:

1. TN: Comments before Ex 7.1 and 9.1; WW: 10.3.4
2. TN: Comments before Ex 7.1; PP: #1-9
3. TN: Ex 15.4 and 15.5; WW: 10.7.1
4. TN: Ex 4.9 and 5.12, comments after Ex 5.14; WW: all of 10.6; PP: #39