From Lecture (4-24-19)

Find \( \iint_R \mathbf{F} \cdot \mathbf{n} \, dS \) where \( \mathbf{F}(x,y,z) = \langle x, -2z, y + z \rangle \) and the surface is the plane \( 3x + 3y + z = 6 \) contained in the first octant, with positive flow being in the positive \( z \) direction.

- Write the surface in explicit form. We can isolate \( z \): \( z = 6 - 3x - 3y \).
- Parameterize the surface: \( \mathbf{r}(x,y) = \langle x, y, 6 - 3x - 3y \rangle \).
- Find partial derivatives: \( \mathbf{r}_x = \langle 1, 0, -3 \rangle \) and \( \mathbf{r}_y = \langle 0, 1, -3 \rangle \).
- Find the normal vector: \( \mathbf{n} = \mathbf{r}_x \times \mathbf{r}_y = \langle 3, 3, 1 \rangle \).
  - Check that this cross product is orthogonal to the originating vectors.
  - Note that \( \mathbf{n} \) “should be” a unit vector, but that the magnitude divisor simplified away when we simplified the original flux integral.
  - Also, note that the \( z \) component should have a 1, to match with “positive flow”.
- Find \( \mathbf{F} \cdot \mathbf{n} \) and simplify:
  \[
  \mathbf{F} \cdot \mathbf{n} = \langle x, -2z, y + z \rangle \cdot \langle 3, 3, 1 \rangle \\
  = 3x - 6z + y + z \\
  = 3x + y - 5z \\
  = 3x + y - 5(6 - 3x - 3y) \\
  = 18x + 16y - 30.
  \]
- Integrate over \( R \):
  \[
  \int_0^2 \int_0^{2-x} (18x + 16y - 30) \, dy \, dx.
  \]
  The inner integral is
  \[
  \int_0^{2-x} (18x + 16y - 30) \, dy = [18xy + 8y^2 - 30y]_0^{2-x} \\
  = 18x(2 - x) + 8(2 - x)^2 - 30(2 - x) \\
  = -10x^2 + 34x - 28.
  \]
  The outer integral is
  \[
  \int_0^2 (-10x^2 + 34x - 28) \, dx = \left[ -\frac{10}{3}x^3 + 17x^2 - 28x \right]_0^2 \\
  = -\frac{80}{3} + 68 - 56 \\
  = -\frac{44}{3}.
  \]
Note that we can solve the surface for $x$ or $y$. Let's see what happens when we solve for $y$. We declare that positive plow is in the direction of positive $y$. Remember, $\mathbf{F}(x, y, z) = \langle x, -2z, y + z \rangle$.

- We have $y = 2 - x - \frac{1}{3}z$.
- In parameter form, we have $\mathbf{r}(x, z) = \langle x, 2 - x - \frac{1}{3}z, z \rangle$.
- The cross product results in $\mathbf{n} = \mathbf{r}_x \times \mathbf{r}_z = \langle 1, 1, \frac{1}{3} \rangle$.
- Find $\mathbf{F} \cdot \mathbf{n}$ and simplify:
  
  \[
  \mathbf{F} \cdot \mathbf{n} = \langle x, -2z, y + z \rangle \cdot \langle 1, 1, \frac{1}{3} \rangle \\
  = x - 2z + \frac{1}{3}y + \frac{1}{3}z \\
  = x - 2z + \frac{1}{3}(2 - x - \frac{1}{3}z) + \frac{1}{3}z \\
  = \frac{2}{3}x - \frac{16}{9}z + \frac{2}{3}.
  \]

- Integrate over $R$:
  
  \[
  \int_0^2 \int_0^{6-3x} \left( \frac{2}{3}x - \frac{16}{9}z + \frac{2}{3} \right) dz \, dx.
  \]

  The inner integral is
  
  \[
  \int_0^{6-3x} \left( \frac{2}{3}x - \frac{16}{9}z + \frac{2}{3} \right) dz = \left[ \frac{2}{3}xz - \frac{8}{9}z^2 + \frac{2}{3}z \right]_0^{6-3x} \\
  = \frac{2}{3}x(6 - 3x) - \frac{8}{9}(6 - 3x)^2 + \frac{2}{3}(6 - 3x) \\
  = -10x^2 + 34x - 28.
  \]

  The outer integral is
  
  \[
  \int_0^2 (-10x^2 + 34x - 28)dx = \left[ -\frac{10}{3}x^3 + 17x^2 - 28x \right]_0^2 \\
  = -\frac{80}{3} + 68 - 56 \\
  = -\frac{44}{3}.
  \]

We get the same value. It does not matter if we set it up as a $dy \, dx$, $dz \, dx$ or $dz \, dy$ integral.
Let’s now look at the flow of $\mathbf{F}(x,y,z) = (x,-2z,y+z)$ through the entire solid, the four-sided object (tetrahedron) composed of the planes $3x + 3y + z = 6$ and the $xy$-plane, the $yz$-plane and the $xz$-plane.

Positive flow through a simple object in $R^3$ is always outward from the interior. A simple object has a distinct inside and outside, with no interior voids or other odd behavior.

From the previous two pages, we showed the flow of $\mathbf{F}$ through the plane $3x + 3y + z = 6$ is $\frac{-44}{3}$.

- For the $xy$-plane, we have $z = 0$, so that $\mathbf{F}(x,y,z) = (x,0,y)$ and $\mathbf{n} = (0,0,-1)$ (point away from the interior)
  - $\mathbf{F} \cdot \mathbf{n} = -y$.
  - The flux is $\int_0^2 \int_0^{2-x} (-y) \, dy \, dx$.
  - The inner integral is $\left[-\frac{1}{2} y^2\right]_0^{2-x} = -\frac{1}{2} (2-x)^2$.
  - The outer integral is $\int_0^2 \left(-\frac{1}{2} (2-x)^2\right) \, dx = \left[\frac{1}{6} (2-x)^3\right]_0^2 = \left(0 - \frac{8}{6}\right) = -\frac{4}{3}$.

- For the $xz$-plane, we have $y = 0$, so that $\mathbf{F}(x,y,z) = (x,-2z,z)$ and $\mathbf{n} = (0,-1,0)$ (point away from the interior)
  - $\mathbf{F} \cdot \mathbf{n} = 2z$.
  - The flux is $\int_0^2 \int_0^{6-3x} (2z) \, dz \, dx$.
  - The inner integral is $\left[z^2\right]_0^{6-3x} = (6-3x)^2$.
  - The outer integral is $\int_0^2 (6-3x)^2 \, dx = \left[\frac{1}{9} (6-3x)^3\right]_0^2 = \left(0 - \left(-\frac{216}{9}\right)\right) = 24$.

- For the $yz$-plane, we have $x = 0$, so that $\mathbf{F}(x,y,z) = (0,-2z,y+z)$ and $\mathbf{n} = (-1,0,0)$ (point away from the interior)
  - $\mathbf{F} \cdot \mathbf{n} = 0$.
  - The flux is $\int_0^2 \int_0^{6-3y} (0) \, dz \, dy = 0$.

The total flux through the solid is $\frac{-44}{3} + \left(-\frac{4}{3}\right) + 24 = 8$. There is net positive outward flow through this solid.

Using the divergence theorem, we have $\text{div } \mathbf{F} = 1 + 0 + 1 = 2$, so that we have

$$\int_R \mathbf{F} \cdot \mathbf{n} \, dS = \iiint_S 2 \, dV.$$ 

In this case, the 2 can be factored to the front, and $\iiint_S dV$ is the volume of the tetrahedron, which is $V = \frac{1}{3} (\text{area base})(\text{height}) = \frac{1}{3} \left(\frac{1}{2} (2)(2)\right) (6) = 4$. Thus, we have

$$\int_R \mathbf{F} \cdot \mathbf{n} \, dS = \iiint_S 2 \, dV = 2 \iiint_S dV = 2(4) = 8.$$