\[ F = \langle 2x, 3 \rangle \]

Find \( \int_C F \cdot dr \) \ for \( C \): line from \((1,0)\) to \((0,1)\)

Parameterize the path \( C \):
\[
\begin{align*}
\mathbf{r}(t) &= \langle 1-t, t \rangle \quad 0 \leq t \leq 1 \\
y(t) &= 0 + t
\end{align*}
\]

\[
\hat{r}'(t) = \langle -1, 1 \rangle \\
\hat{r}'(t) = \langle 2(1-t), 3 \rangle = \langle 2-2t, 3 \rangle.
\]

\[
\begin{align*}
\hat{F} \cdot \hat{r}' &= -2 + 2t + 3 \\
\int_0^1 \hat{F} \cdot \hat{r}' \, dt &= \int_0^1 (-2 + 2t + 3) \, dt \\
&= \left[-2t + t^2 + t\right]_0^1 = 2
\end{align*}
\]

\( C \): quarter circle, \( r = 1 \), centered at \((0,0)\) from \((1,0)\) to \((0,1)\)

\[
\begin{align*}
x(t) &= \cos t \\
y(t) &= \sin t
\end{align*}
\]

\[
\hat{r}(t) = \langle \cos t, \sin t \rangle, \quad \hat{r}'(t) = \langle -\sin t, \cos t \rangle
\]

\[
\hat{F}(\hat{r}(0)) = \langle 2 \cos t, 3 \rangle \\
\hat{F} \cdot \hat{r}' = -2 \sin t \cos t + 3 \cos t
\]

\[
\int_0^{\pi/2} (-2 \sin t \cos t + 3 \cos t) \, dt = (-\sin^2 t + 3 \sin t) \bigg|_0^{\pi/2} = (-1) - (-0) = 2
\]

For \( \hat{F}(x,y) = \langle 2x, 3 \rangle \),
the value of \( \int F \cdot dr \) is independent of the actual path.
\( \hat{F} \) is "conservative" ("gradient" field)

Assume that \( \hat{F}(x,y) = \langle P(x,y), Q(x,y) \rangle \)

If \( P_y = Q_x \), then \( \hat{F} \) is conservative.

Then there exists some "potential function" \( f(x,y) \), such that \( \nabla f = \langle P(x,y), Q(x,y) \rangle \).

\[
\begin{align*}
f_x &= 2x \\
f_y &= 3
\end{align*}
\]

For \( \hat{F}(x,y) = \langle 2x, 3 \rangle \), what \( f(x,y) \) "generates" this vector field?
We need \( f_x = 2x \) and \( f_y = 3 \) \( \Rightarrow f(x,y) = x^2 + 3y \).

Test it: \( \nabla f = \langle 2x, 3 \rangle \) This works.
Suppose we start with a function \( f(x,y) \).
We form a vector field from \( f \) by its gradient: \( \nabla f = \langle f_x, f_y \rangle = \vec{F} \).
Suppose we parameterize a path \( \vec{r}(t) = \langle x(t), y(t) \rangle \).
So, therefore, \( \vec{r}'(t) = \langle \frac{dx}{dt}, \frac{dy}{dt} \rangle \).
We calculate \( \int \vec{F} \cdot d\vec{r} = \int \left( f_x \, dx + f_y \, dy \right) \). 
From Ch. 11, we have a chain rule: \( \frac{d}{dt} \vec{F}(\vec{r}(t)) = \frac{df}{dx} \frac{dx}{dt} + \frac{df}{dy} \frac{dy}{dt} \).
So, \( \int \vec{F} \cdot d\vec{r} = \int \frac{d}{dt} \vec{F}(\vec{r}(t)) \bigg|_{t_0}^{t_1} = \vec{F}(\vec{r}(t_1)) - \vec{F}(\vec{r}(t_0)) \). 

Recall: \( \vec{F}(x,y) = \langle 2x, 3 \rangle \)
We were parameterizing paths from \((1,0)\) to \((0,1)\).
We showed that \( \vec{F} \) was conservative \( (P_y = Q_x) \).
We found the potential function, \( f(x,y) = x^2 + 3y \).
\( \int \vec{F} \cdot d\vec{r} = (x^2 + 3y) \bigg|_{(1,0)}^{(0,1)} = (0+3) - (1+0) = 2 \).

Let \( \vec{F} = \langle 3x^2 y, 2x^3 y \rangle \) be a parabola \( y = x^2 \) from \((1,1)\) to \((3,9)\).
Find \( \int \vec{F} \cdot d\vec{r} \):
1. Is \( \vec{F} \) conservative? \( P_y = Q_x \)? \( P_y = 6x^2 y \quad Q_x = 6x^2 y \quad \text{Yes!} \)
2. Find the potential function \( f(x,y) \): \( \begin{cases} 3x^2 y - 6x^2 y = x^2 y^2 - 2x^3 y^2 \\ 2x^3 y + 2x^3 y = x^3 y^2 \end{cases} \) \( f(x,y) = x^3 y^2 \).
3. \( \int \vec{F} \cdot d\vec{r} = x^3 y^2 \bigg|_{(1,1)}^{(3,9)} = 27 \cdot 81 - 1 = 2186 \).
Corollary:

If \( \vec{F} \) is a conservative vector field, and
the path is any simple, closed, positively oriented (loop),
then \( \int_C \vec{F} \cdot d\vec{r} = 0 \) (because you have the same start/end pt).

Simple: does not cross itself.

ex: \( \bigcirc \) is "simple" \( \bigcirc \) is not.

closed: starts and ends at the same point.

positively oriented: \( \text{Counter clockwise} \) is "positive"

\( \bigcirc \) positively oriented \( \bigcirc \) negatively oriented

\( \vec{F} \) is conservative?

<table>
<thead>
<tr>
<th>( \text{Yes} )</th>
<th>( \text{No} )</th>
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Is \( C \) (the path) a loop?

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<thead>
<tr>
<th>( \text{Yes} )</th>
<th>( \text{No} )</th>
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</thead>
<tbody>
<tr>
<td>Use the F.T.L.I.</td>
<td>( \text{You do it &quot;the long way&quot;, by parameterizing, etc.} )</td>
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</tbody>
</table>

Use Green's Theorem (next section)