Scalar: Let \( z = f(x, y) \) be a surface and let \( \mathbf{r}(t) = \langle x(t), y(t) \rangle \), as \( a \leq t \leq b \), be a path in the \( xy \)-plane. The area of the "curtain" is

\[
\int_a^b f(x, y) \, ds,
\]

where \( ds = |\mathbf{r}'(t)| \, dt \).

Process:
1. Parameterize the path in variable \( t \).
2. Convert \( f(x, y) \) into \( t \).
3. Replace \( ds \) with \( |\mathbf{r}'(t)| \, dt \).
4. Integrate.

**Example:** Find the area below \( z = xy^2 + 2x^3 \) over the line segment from \((1,2)\) to \((4,8)\).

1. Parameterize the line: \( \mathbf{r}(t) = \left( \frac{1+3t}{x}, \frac{2+6t}{y} \right) \) \( 0 \leq t \leq 1 \)

2. Convert \( F \) in terms of \( t \):
   \[
   f = xy^2 + 2x^3 = \frac{(1+3t)(2+6t)^2 + 2(1+3t)^3}{(1+3t)(2+6t)^2 + 2(1+3t)^3} \]
   \[
   \int_0^1 \sqrt{(1+3t)(2+6t)^2 + 2(1+3t)^3} \, dt
   \]
   (Foil out or use u-sub substitution...)

3. \( \mathbf{r}' = (3,6) \), so \( |\mathbf{r}'(t)| = \sqrt{45} \)

**Example:** Find the area below \( z = 2xy \) over the curve \( y = x^2 \) from \((0,0)\) to \((2,4)\).

1. Parameterize: \( \mathbf{r}(t) = \left( \frac{t}{x}, \frac{t^2}{y} \right) \) \( 0 \leq t \leq 2 \)

2. Convert integrand into \( t \):
   \[
   f = 2xy = 2(t)(t^2) = 2t^3
   \]
   \[
   \int_0^2 2t^3 \sqrt{1+4t^2} \, dt
   \]
   (on hw, use your calculator)

3. \( \mathbf{r}' = (1,2t) \), so \( |\mathbf{r}'(t)| = \sqrt{1+4t^2} \)
Work Like Integrals

Let \( \vec{F}(x,y) = \langle M(x,y), N(x,y) \rangle \) be a vector field in \( \mathbb{R}^2 \) and \( c: \vec{r}(t) = \langle x(t), y(t) \rangle \) be a path. An object moves along the path where as \( t \in \mathbb{R} \). At each point on the path, the object's direction is given by \( \vec{r}'(t) = \langle x'(t), y'(t) \rangle \), and there is a force \( \vec{F}(x,y) \) acting on the object.

The work done by \( \vec{F} \) on the object is \( \vec{F} \cdot \vec{r}' \) at that point.

**Problem:**
- Object may speed up or slow down because \( \vec{r}'(t) \) varies in length.

**Solution:**
- Use the unit tangent vector \( \vec{T}(t) = \frac{\vec{r}'(t)}{|\vec{r}'(t)|} \)

Thus, the total work done is the "sum" of the work at each point:

\[
\int_0^1 \vec{F} \cdot \vec{T} \, ds = \int_0^1 \frac{\vec{F}(t)}{|\vec{r}'(t)|} \cdot |\vec{r}'(t)| \, dt
\]

**Simplify:** \( \int_0^1 \vec{F} \cdot \vec{T} \, ds = \int_0^1 \vec{F}(t) \cdot \vec{T}(t) \, dt \)

**Other common forms:**
- \( \int_c \vec{F} \cdot ds \), \( \int_c M \, dx + N \, dy \)

**Ex:** Find the work done on a particle moving in a straight line from \((2,3)\) to \((6,1)\), in vector field \( \vec{F}(x,y) = \langle 3x^2, x+y \rangle \)

1. Parameterize the path: \( \vec{r}(t) = \langle 2+4t, 3-2t \rangle \), \( 0 \leq t \leq 1 \)
2. Write \( \vec{F} \) in terms of \( t \): \( \vec{F}(t) = \langle 3(t+4t)^2, (2+4t) + (3-2t) \rangle = \langle 12+48t+48t^2, 5+2t \rangle \)
3. \( \vec{r}' = \langle 4, -2 \rangle \)
4. \( \vec{F} \cdot \vec{r}' = 4(12+48t+48t^2) + (-2)(5+2t) = 48+192t+192t^2-10-4t = 192t^2+188t+38 \), \( \text{pos. quantity} \)
5. \( \int_0^1 192t^2+188t+38 \, dt = \frac{64t^3+94t^2+38t}{196} \bigg|_0^1 = 64+94+38 \) in the direction of movement.
Ex: Find the work done by \( \vec{F}(x,y) = \langle x^2y, 2y \rangle \) on a particle in path \( C \) which is a circular arc from \((2,0)\) to \((0,2)\).

1. Parameterization: \( \vec{r}(t) = \langle 2\cos t, 2\sin t \rangle \) \( 0 \leq t \leq \frac{\pi}{2} \)

2. Substitution: \( \vec{F}(t) = \langle (2\cos t)^2(2\sin t), 2(2\sin t) \rangle = \langle 8\cos^2 t \sin t, 4 \sin t \rangle \)

3. \( d\vec{r} = \langle -2\sin t, 2\cos t \rangle \)

4. Dot: \( \vec{F} \cdot d\vec{r} = -16 \cos^2 t \sin^2 t + 8 \cos t \sin t \)

5. Integral: \( \int_0^{\frac{\pi}{2}} \left[ -16 \cos^2 t \sin^2 t + 8 \cos t \sin t \right] dt \)

   Use half-angle identities here.