\[ \int_0^6 \int_0^{\sqrt{x}} (2x+y) \, dy \, dx + \int_6^9 \int_{x-6}^{\sqrt{x}} (2x+y) \, dy \, dx \]

rewrite as a single double-integral in \( dx \, dy \)

\[ \int_0^3 \int_{y^2}^{y+6} (2x+y) \, dx \, dy \]

\[ \int_0^1 \int_0^{\sqrt{9-x^2}} \frac{\sqrt{9-x^2}}{\sqrt{1-x^2}} \, dy \, dx + \int_1^3 \int_0^{\sqrt{9-x^2}} \frac{\sqrt{9-x^2}}{\sqrt{x^2+y^2}} \, dy \, dx \]

\[ \text{Use polar: } \begin{cases} 1 \leq r \leq 3 \\ 0 \leq \theta \leq \frac{\pi}{2} \end{cases} \}

\[ \frac{1}{3} \int_0^{\pi/2} \int_1^3 r^2 \, dr \, d\theta = \frac{2}{3} \int_0^{\pi/2} \int_0^3 r^2 \, dr \, d\theta = \frac{2}{3} \int_0^{\pi/2} \frac{1}{3} r^3 \, d\theta = \frac{26}{6} \pi = \frac{13\pi}{3} \]
Find the volume below \( z = 9 - x^2 - y^2 \) over the xy plane.

\[
9 - (x^2 + y^2)
\]

\[
2\pi \int_0^3 \int_0^{\sqrt{9-r^2}} (9-r^2) r \, dr \, d\theta
\]

Circle, \( x^2 + y^2 = 9 \)

12.4 Masses, Center of mass.

Easy case: you have a geometric shape with uniform density, for example: a rectangle that is 3 in \( \times \) 5 in, and density is 2 lb/\( \text{in}^2 \).

\[
15 \text{ in}^2 \times 2 \text{ lb/\text{in}}^2 = 30 \text{ lbs.}
\]

Suppose the density is variable. For example, suppose the population density \( \rho \) of a town is given by \( \rho(x,y) = 10 + x + 2y \), where \( 0 \leq x \leq 2 \), \( 1 \leq y \leq 3 \), and \( x \leq y \) are in miles. \( \rho \) in thousands of people/mile \(^2 \).

\[
\text{Pop} = \int_0^2 \int_1^3 (10 + x + 2y) \, dy \, dx
\]

Center of mass.

* 2 objects of equal weight, one at \( x = 3 \), one at \( x = 5 \). Where's the center of mass? \( x = 4 \).

* 2 objects at \( x = 3 \), one at \( x = 5 \) (all weight the same). Center of mass? \( \frac{2(3) + 1(5)}{3} = \frac{11}{3} = 3\frac{2}{3} \).

* Let \( f(x) \) be the "weight" at an \( x \)-value. To find the average value of \( f(x) \) over an \( x \)-interval,

\[
\text{Sum of } x-\text{values times their function values} \quad \Sigma x f(x) = \int x f(x) \, dx \quad \bar{x}
\]

\[
\int f(x) \, dx
\]
Town, population density is \( p(x, y) = (10x + 2y) \) for \( 0 \leq x \leq 2, \ 1 \leq y \leq 3 \).

Where's the center of population?

\[
P = \int_0^2 \int_1^3 (10x + 2y) \, dy \, dx
\]

\[
= \int_0^2 \left[ 10y + xy + y^2 \right]_1^3 \, dx
\]

\[
= 30 + 3x + 9 - (10 + x + 1)
\]

\[
= 39 + 3x - 11 - x
\]

\[
= 28 + 2x
\]

\[
\int_0^2 28 + 2x \, dx = 28x + x^2 \bigg|_0^2 = 56 + 4 = 60 \quad \text{(about 60,000)}
\]

Coordinates of the center:

\[
\bar{x} = \frac{\int_0^2 \int_1^3 x \, p(x, y) \, dy \, dx}{\int_0^2 \int_1^3 p(x, y) \, dy \, dx}
\]

\[
\bar{y} = \frac{\int_0^2 \int_1^3 y \, p(x, y) \, dy \, dx}{\int_0^2 \int_1^3 p(x, y) \, dy \, dx}
\]

\(
\bar{x} \quad \text{(top)}: \quad \int_0^2 \int_1^3 x(10x + 2y) \, dy \, dx
\)

\[
= \int_0^2 \left[ 10x^2 + 2xy \right]_1^3 \, dx
\]

\[
= 30x + 3x^2 + 8x - (10x + x^2 + x)
\]

\[
= 20x + 2x^2 + 8x
\]

\[
\int_0^2 20x + 2x^2 + 8x \, dx = \left[ 10x^2 + \frac{2}{3}x^3 + 4x^2 \right]_0^2
\]

\[
= 40 + \frac{16}{3} + 16 \approx 61.333...
\]

\[
\bar{x} = \frac{61.333...}{60} = 1.02...
\]

\( \bar{y} \) works out similarly, will probably be slightly larger than 2.