Lecture, 1-30-14

Lines and planes in $\mathbb{R}^3$.

**Line:**
- A point $(x_0, y_0, z_0)$
- A vector $\vec{v} = \langle v_1, v_2, v_3 \rangle$ (parallel to line)

$$
\begin{align*}
x &= x_0 + tv_1 \\
y &= y_0 + tv_2 \\
z &= z_0 + tv_3
\end{align*}
$$

**Plane:**
- A point $(x_0, y_0, z_0)$
- A vector $\vec{n} = \langle a, b, c \rangle$ (orthogonal to the plane) — "normal vector"

$$
ax + by + cz = d, \text{ when } d = ax_0 + by_0 + cz_0
$$

**Example:**

**P:** $(1, 2, 1)$  
**Q:** $(3, -1, 0)$  
**R:** $(-4, 2, 3)$.

Find the equation of the plane passing through $P, Q, R$.

Create 2 vectors:

$$
\vec{PQ} = \langle 2, -3, -1 \rangle, \quad \vec{PR} = \langle -5, 0, 2 \rangle
$$

$$
\vec{n} = \vec{PQ} \times \vec{PR} = \begin{vmatrix}
\hat{i} & \hat{j} & \hat{k} \\
2 & -3 & -1 \\
-5 & 0 & 2
\end{vmatrix} = \langle 12, -10, 1 \rangle = \langle 3, -2, 1 \rangle
$$

$$
-6x + y - 15z = -19
$$

To get the constant, plug in any one of the points. Check w/ the others.

**Example:** 2 planes:

$P_1: x + 2y + 3z = 5 \implies \vec{n}_1 = \langle 1, 2, 3 \rangle$

$P_2: 3x - y + 2z = 1 \implies \vec{n}_2 = \langle 3, -1, 2 \rangle$

**Q:** What angle do these planes intersect?

$$\theta = \cos^{-1} \left( \frac{\vec{n}_1 \cdot \vec{n}_2}{\|\vec{n}_1\| \|\vec{n}_2\|} \right) = \cos^{-1} \left( \frac{7}{\sqrt{14} \sqrt{14}} \right) = \cos^{-1} \left( \frac{1}{2} \right) = 60^\circ$$
Ex 2, continued.

Q: What is the equation of the line at which the planes intersect?

Note: the line of intersection is orthogonal to the 2 normals...

So cross them:

\[
\mathbf{v} = \mathbf{n}_1 \times \mathbf{n}_2 = \begin{vmatrix} i & j & k \\ 1 & 2 & 3 \\ 2 & 3 & 1 \end{vmatrix} = (-1)2i - 13j + 1 \mathbf{k} = (7, 7, -7)
\]

we need a point... go back to the 2 planes:

\[
\begin{align*}
x + 2y + 3z &= 5 \\
3x - y + 2z &= 1
\end{align*}
\]

Find any point. Example, let \(x = 0\):

\[
\begin{align*}
2y + 3z &= 5 \\
-2y + 2z &= 1
\end{align*}
\]

so \(y = 1\)

the point is \((0, 1, 1)\)...

\[
\begin{align*}
x &= 0 + 7t \\
y &= 1 + 7t \\
z &= 1 + (-7t)
\end{align*}
\]

Ex: Plane: \(2x + y - z = 3\)

and a point \(Q = (3, 5, 2)\). Note: the point is not on the plane.

Q: What's the shortest distance from the point to the plane?

\[
\mathbf{Q} = \begin{pmatrix} 2, 1, -1 \end{pmatrix}
\]

plan: Create a vector \(\mathbf{RQ}\), where \(R\) is on the plane.

then we'll project \(\mathbf{RQ}\) onto \(\mathbf{n}\).

then we'll find the magnitude.

let \(R = (0, 3, 0) \rightarrow \mathbf{RQ} = \begin{pmatrix} 3, 2, 2 \end{pmatrix}\)

\[
\text{proj}_{\mathbf{n}} \mathbf{RQ} = \left( \frac{\mathbf{RQ} \cdot \mathbf{n}}{\mathbf{n} \cdot \mathbf{n}} \right) \mathbf{n} = \left( \frac{6}{8} \right) \mathbf{n} = \frac{3}{4} \mathbf{n}, \quad |\mathbf{n}| = \sqrt{6} \quad \text{ANS.}
\]
10.6 Quadric Surfaces

Common Surfaces we'll see:

* Cylinders. Take a curve that lies in a plane, extend it into the 3rd dimension.

* Spheres & Ellipsoids.
  Ellipsoid: \(2x^2 + 3y^2 + z^2 = 9\)

* Paraboloid:
  \(z = x^2 + y^2\)
  "paraboloid (like a bowl)"

* Hyperbolic paraboloid:
  \(x^2 - y^2 = z^2\)
  "hyperboloid of one sheet"
  "nuclear plant cooling tower"

* Hyperboloid of two sheets:
  \(x^2 - y^2 = z^2\)
  "hyperboloid of 2 sheets."

* Elliptic paraboloid:
  \(z = x^2 - y^2\)
  "saddle" parabolic hyperboloid

See your text for a table of common shapes.