49. Green’s Theorem

Let \( \mathbf{F}(x,y) = (M(x,y), N(x,y)) \) be a vector field in \( R^2 \), and suppose \( C \) is a path that starts and ends at the same point such that it does not cross itself. Such a path is called a simple closed loop, and it will enclose a region \( R \). Assume \( M \) and \( N \) and its first partial derivatives are defined within \( R \) including its boundary \( C \). Furthermore, the path is to be traversed (circulated) in a counterclockwise direction, called the positive orientation. If these conditions are met, then the line integral around the simple loop path may be evaluated by a double integral. This is called Green’s Theorem, and is written

\[
\int_C \mathbf{F} \cdot d\mathbf{r} = \iint_R (N_x - M_y) \, dA.
\]

If \( \mathbf{F} \) is a conservative vector field, then \( M_y = N_x \), so that the integrand \( N_x - M_y = 0 \). Thus, in a conservative vector field, all line integrals along a simple closed loop path evaluate to 0. In a physical sense, there is no net circulation around the loop, and a conservative vector field is often called a rotation-free (or irrotational) vector field.

When calculating a line integral, you should check two things:

- Is the vector field conservative?
- Is the path a simple closed loop?

The following table will help you plan your calculation accordingly.

<table>
<thead>
<tr>
<th></th>
<th>( \mathbf{F} ) is conservative</th>
<th>( \mathbf{F} ) is not conservative</th>
</tr>
</thead>
<tbody>
<tr>
<td>( C ) is a simple closed loop</td>
<td>0</td>
<td>Use Green’s Theorem</td>
</tr>
<tr>
<td>( C ) is not a loop of any kind (it has different start and end points).</td>
<td>Find the potential function ( \phi(x,y) ) and calculate the line integral by the Fundamental Theorem of Line Integrals (The FTLI)</td>
<td>Parameterize the path(s) in variable ( t ), and calculate the line integral directly.</td>
</tr>
</tbody>
</table>
**Example 49.1:** Evaluate \( \int_C \mathbf{F} \cdot d\mathbf{r} \), where \( \mathbf{F}(x, y) = \langle y, 4x \rangle \) and \( C \) is a triangle, traversed from \((0,0)\) to \((2,0)\) to \((2,4)\) back to \((0,0)\).

**Solution:** We sketch \( C \) and verify that it is a simple closed loop that is traversed counter-clockwise:

To evaluate \( \int_C \mathbf{F} \cdot d\mathbf{r} \) as a sequence of line integrals, we would need to divide the path into three smaller paths: \( C_1 \) being the line from \((0,0)\) to \((2,0)\), \( C_2 \) being the line from \((2,0)\) to \((2,4)\), and \( C_3 \) being the line from \((2,4)\) to \((0,0)\).

- For \( C_1 \), we have \( \mathbf{r}_1(t) = \langle 2t, 0 \rangle \) with \( 0 \leq t \leq 1 \), so that \( \mathbf{r}_1'(t) = \langle 2, 0 \rangle \) and \( \mathbf{F}(t) = \langle 0, 0, 0 \rangle \). Thus, in this case, \( \mathbf{F} \cdot d\mathbf{r}_1 = 0 \), so that \( \int_{C_1} \mathbf{F} \cdot d\mathbf{r}_1 = 0 \). In the above image, note that the vector field elements are orthogonal to the segment \( C_1 \).

- For \( C_2 \), we have \( \mathbf{r}_2(t) = \langle 2t, 4t \rangle \) with \( 0 \leq t \leq 1 \), so that \( \mathbf{r}_2'(t) = \langle 0, 4 \rangle \) and \( \mathbf{F}(t) = \langle 4t, 8t \rangle \). Thus, \( \mathbf{F} \cdot d\mathbf{r}_2 = \langle 4t, 8t \rangle \cdot \langle 0, 4 \rangle = 32 \), so that \( \int_{C_2} \mathbf{F} \cdot d\mathbf{r}_2 = \int_0^1 32 \, dt = [32t]_0^1 = 32 \). The vector field elements agree with the direction of \( C_2 \).

- For \( C_3 \), we have \( \mathbf{r}_3(t) = \langle 2 - 2t, 4 - 4t \rangle \) with \( 0 \leq t \leq 1 \), so that \( \mathbf{r}_3'(t) = \langle -2, -4 \rangle \) and \( \mathbf{F}(t) = \langle 4 - 4t, 8 - 8t \rangle \). Thus, \( \mathbf{F} \cdot d\mathbf{r}_3 = \langle 4 - 4t, 8 - 8t \rangle \cdot \langle -2, -4 \rangle = 40t - 40 \), so that \( \int_{C_3} \mathbf{F} \cdot d\mathbf{r}_3 = \int_0^1 (40t - 40) \, dt = [20t^2 - 40t]_0^1 = -20 \). The vector field elements disagree (point against) the direction of \( C_3 \).

Since

\[
\int_C \mathbf{F} \cdot d\mathbf{r} = \int_{C_1} \mathbf{F} \cdot d\mathbf{r}_1 + \int_{C_2} \mathbf{F} \cdot d\mathbf{r}_2 + \int_{C_3} \mathbf{F} \cdot d\mathbf{r}_3 ,
\]

we have

\[
\int_C \mathbf{F} \cdot d\mathbf{r} = 0 + 32 - 20 = 12.
\]

282
Now, let’s use Green’s Theorem. We find that \(N_x - M_y = 4 - 1 = 3\), so that

\[
\int_C \mathbf{F} \cdot d\mathbf{r} = \iint_R 3 \, dA
\]
\[
= 3 \iint_R dA
\]
\[
= 3(4)
\]
\[
= 12.
\]

In this case, the constant integrand was moved to the front, leaving \(\iint_R dA\), which is the area of region \(R\). Using geometry, the area of \(R\) is that of a triangle with base 2 and height 4, so \(\iint_R dA = \frac{1}{2} (2)(4) = 4\).

Example 49.2: Evaluate \(\int_C \mathbf{F} \cdot d\mathbf{r}\), where \(\mathbf{F}(x, y) = (2xy, x)\) and \(C\) traverses from \((2,0)\) to \((-2,0)\) along a semi-circle of radius 2, centered at the origin, in the counter-clockwise direction, then from \((-2,0)\) back to \((2,0)\) along a straight line.

**Solution:** Path \(C\) is a simple closed loop traversed in a counter-clockwise direction.

To find \(\int_C \mathbf{F} \cdot d\mathbf{r}\), we use Green’s Theorem. Since the region \(R\) is a semicircle of radius 2, we will evaluate the double integral using polar coordinates.

\[
\int_C \mathbf{F} \cdot d\mathbf{r} = \iint_R (N_x - M_y) \, dA
\]
\[
= \iint_R (1 - 2x) \, dA
\]
\[
= \int_0^\pi \int_0^2 (1 - 2r \cos \theta) \, r \, dr \, d\theta
\]
\[
= \int_0^\pi \int_0^2 (r - 2r^2 \cos \theta) \, dr \, d\theta.
\]
The inside integral is evaluated with respect to $r$:

$$\int_{0}^{r} (r - 2r^2 \cos \theta) \, dr = \left[ \frac{1}{2} r^2 - \frac{2}{3} r^3 \cos \theta \right]_{0}^{2} = 2 - \frac{16}{3} \cos \theta.$$

This is then integrated with respect to $\theta$:

$$\int_{0}^{\pi} \left( 2 - \frac{16}{3} \cos \theta \right) \, d\theta = \left[ 2\theta - \frac{16}{3} \sin \theta \right]_{0}^{\pi} = 2\pi.$$

Thus, the line integral along $C$ is $\int_{C} \mathbf{F} \cdot d\mathbf{r} = 2\pi$. There is positive circulation along this path induced by the vector field.

Example 49.3: Evaluate $\int_{C} \mathbf{F} \cdot d\mathbf{r}$, where $F(x, y) = \langle 3y, -x + y \rangle$ and $C$ traverses a rectangle from $(1,1)$ to $(1,6)$ to $(7,6)$ to $(7,1)$ back to $(1,1)$.

Solution: A sketch of the path $C$ shows it to be a simple closed loop traversed in a clockwise direction. In order to use Green’s Theorem, we would traverse it in the counter-clockwise direction, which is equivalent to traversing each segment in its opposite direction. This means that we will multiply our result by $-1$ in order to account for this “opposite” direction.

Using Green’s Theorem, we have $N_x - M_y = -1 + (-3) = -4$:

$$\iint_{R} (N_x - M_y) \, dA = \iint_{R} (-4) \, dA = -4 \int_{1}^{7} \int_{1}^{6} \, dy \, dx.$$

The double integral $\int_{1}^{7} \int_{1}^{6} \, dy \, dx$ is the area of the rectangle, which is $(6)(5) = 30$. Thus,

$$\iint_{R} (N_x - M_y) \, dA = -4(30) = -120.$$

However, since $C$ was traversed in the opposite direction, we negate this result. We have

$$\int_{C} \mathbf{F} \cdot d\mathbf{r} = 120.$$
Example 49.4: Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F}(x, y) = (5x^4 + y^2, 2yx)$ and $C$ is an ellipse with major axis of 12 along the $x$-axis, and minor axis of 8 along the $y$-axis, in a counter-clockwise direction.

Solution: Using Green’s Theorem, we have

$$\iint_R (N_x - M_y) \, dA = \iint_R (2y - 2y) \, dA = \iint_R 0 \, dA = 0.$$ 

Note that $\mathbf{F}$ is conservative, since $M_y = N_x$. There is no need to parameterize the ellipse.

Green’s Theorem can be used to find the line integral of a non-loop path. We “close off” the path forming a loop, as this next example shows:

Example 49.5: Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F}(x, y) = (2y, x^2)$ and $C$ is a sequence of line segments from $(0,0)$ to $(3,0)$ to $(3,4)$ to $(-4,4)$.

Solution: The path $C$ is not a simple closed loop. Thus, we would have to parameterize each line segment one at a time and determine the value of each line integral individually. Instead, we can add in the final line segment, that from $(-4,4)$ to $(0,0)$, thus creating a simple closed loop traversed counter-clockwise.

Using Green’s Theorem, we have

$$\iint_R (N_x - M_y) \, dA = \int_0^3 \int_{-y}^3 (2x - 2y) \, dx \, dy = 2 \int_0^3 \int_{-y}^3 (x - 1) \, dx \, dy.$$
The inside integral is first evaluated:

\[
\int_{-y}^{3} (x - 1) \, dx = \left[ \frac{1}{2} x^2 - x \right]_{-y}^{3} = \left( \frac{9}{2} - 3 \right) - \left( \frac{1}{2} y^2 + y \right) = -\frac{1}{2} y^2 - y + \frac{3}{2}.
\]

This result is then integrated with respect to \( y \):

\[
2 \int_{0}^{4} \left( -\frac{1}{2} y^2 - y + \frac{3}{2} \right) \, dy = \left[ -\frac{1}{3} y^3 - y^2 + 3y \right]_{0}^{4} = -\frac{76}{3}.
\]

We now need to evaluate the line integral from \((-4,4)\) to \((0,0)\), the segment that we added in to form the closed loop.

We have \( \mathbf{r}(t) = (-4 + 4t, 4 - 4t) \), where \( 0 \leq t \leq 1 \). Thus, \( \mathbf{r}'(t) = (4, -4) \) and \( \mathbf{F}(t) = \langle 2(4 - 4t), (-4 + 4t)^2 \rangle \), which simplifies to \( \mathbf{F}(t) = \langle 8 - 8t, 16t^2 - 32t + 16 \rangle \). Along this path segment, we have

\[
\int_{C} \mathbf{F} \cdot d\mathbf{r} = \int_{0}^{1} \langle 8 - 8t, 16t^2 - 32t + 16 \rangle \cdot (4, -4) \, dt
\]

\[
= \int_{0}^{1} (-64t^2 + 96t - 32) \, dt
\]

\[
= \left[ -\frac{64}{3} t^3 + 48t^2 - 32t \right]_{0}^{1}
\]

\[
= -\frac{16}{3}.
\]

Therefore, the line integral from \((0,0)\) to \((3,0)\) to \((3,4)\) to \((-4,4)\) is the value we found from Green’s Theorem, \(-\frac{76}{3}\), subtracted by the value of the line integral along the segment we used to “close off” the region, \(-\frac{16}{3}\). We then have \( \int_{C} \mathbf{F} \cdot d\mathbf{r} = -\frac{76}{3} - \left( -\frac{16}{3} \right) = -\frac{60}{3} = -20. \)
As a check, here are the individual line integrals along the three line segments:

- From (0,0) to (3,0): We have \( \mathbf{r}(t) = \langle 3t, 0 \rangle \) with \( 0 \leq t \leq 1 \). Thus, \( \mathbf{r}'(t) = \langle 3, 0 \rangle \) and \( \mathbf{F}(t) = \langle 0, 9t^2 \rangle \), and therefore \( \int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^1 \langle 0, 9t^2 \rangle \cdot \langle 3, 0 \rangle \, dt = \int_0^1 0 \, dt = 0 \).

- From (3,0) to (3,4): We have \( \mathbf{r}(t) = \langle 3, 4t \rangle \), with \( 0 \leq t \leq 1 \). Thus, \( \mathbf{r}'(t) = \langle 0, 4 \rangle \) and \( \mathbf{F}(t) = \langle 8t, 9 \rangle \), and therefore \( \int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^1 \langle 8t, 9 \rangle \cdot \langle 0, 4 \rangle \, dt = \int_0^1 36 \, dt = [36t]_0^1 = 36 \).

- From (3,4) to (−4,4): We have \( \mathbf{r}(t) = \langle 3 - 7t, 4 \rangle \), with \( 0 \leq t \leq 1 \). Thus, \( \mathbf{r}'(t) = \langle -7, 0 \rangle \) and \( \mathbf{F}(t) = \langle 8, (3 - 7t)^2 \rangle \), and therefore \( \int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^1 \langle 8, (3 - 7t)^2 \rangle \cdot \langle -7, 0 \rangle \, dt = \int_0^1 -56 \, dt = [-56t]_0^1 = -56 \).

The sum is \( 0 + 36 - 56 = -20 \), which agrees with our earlier answer.

If \( C \) is a simple closed loop, then the region \( R \) bounded by \( C \) is **simply connected**. All of the regions in the preceding examples in this section are simply connected. In a very intuitive sense, a simply-connected region in the plane has no holes.

Green’s Theorem requires a simply connected region \( R \). However, a non-simply connected region can be made into two (or more) simply-connected regions by dividing the region carefully.

In the above image, a non-simply connected region is strategically divided into two subregions, \( A \) and \( B \), that are each simply connected. Notice that the counter-clockwise circulation is preserved in both cases. The line integrals along the two “cuts” will cancel, since the flow is in opposite directions depending on whether \( A \) or \( B \) is being considered. Green’s Theorem can then be applied to each subregion, and often combined into one double integral covering the entire region.
Example 49.6: Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F}(x, y) = (e^x + 2y, 3x - \sin y)$ and $C$ is the boundary of a region $R$ enclosed by two concentric circles, centered at the origin, one of radius 5 and the other of radius 3. Assume the circulation in the outer circle is counter-clockwise, and that the circulation on the inner circle is clockwise.

Solution: The region $R$ and its boundary $C$ are shown below.

Using Green’s Theorem, we have $N_x - M_y = 3 - 2 = 1$. Using polar coordinates, the region $R$ can be defined as $3 \leq r \leq 5$ and $0 \leq \theta \leq 2\pi$. Thus,

$$
\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^{2\pi} \int_3^5 r\, dr\, d\theta
$$

$$
= \int_0^{2\pi} \left[ \frac{r^2}{2} \right]_3^5 \, d\theta
$$

$$
= 8 \int_0^{2\pi} \, d\theta \quad \left( \left[ \frac{r^2}{2} \right]_3^5 = 8 \right)
$$

$$
= 8(2\pi) = 16\pi.
$$

Note that $\int_0^{2\pi} \int_3^5 r\, dr\, d\theta$ is the area of $R$ represented as a double integral, so we can verify using geometry. The area inside a circle of radius 5 is $25\pi$, and the area inside a circle of radius 3 is $9\pi$, and their difference is $16\pi$.

© 2009-2016 Scott Surgent (surgent@asu.edu)