Projectile Motion Using Vectors

On Earth, the acceleration vector is always \( a(t) = (0, -9.8) \), where the 0 means there is no acceleration due to gravity in the horizontal direction. All acceleration is “down”, so the vertical component receives a leading negative. The units are \( m/s^2 \).

Integrating, we get a velocity vector,

\[
v(t) = \int a(t) \, dt = (v_1, -9.8t + v_2),
\]

where \( v_1 \) and \( v_2 \) are constants of integration based on an initial velocity vector.

Integrating again, we get the displacement vector,

\[
r(t) = \int v(t) \, dt = (v_1 t + r_1, -4.9t^2 + v_2 t + r_2),
\]

where \( r_1 \) and \( r_2 \) are constants of integration based on an initial displacement vector.

Recall that speed is the magnitude of velocity. Assume negligible air resistance in these problems.

Example: A ball is shot off a 100 m tall cliff at an initial speed of 50 m/s at an angle of 30 degrees with the horizontal.

1) Find the maximum height of the ball.
2) Find how far out horizontally the ball travels before it hits the ground (its “range”)
3) How fast is the ball travelling when it hits the ground below for the first time?
4) At what angle does the ball impact the ground?

Solution: We first build three vector equations that govern the ball’s acceleration, velocity and displacement. We always start with \( a(t) = (0, -9.8) \). Thus, \( v(t) = (v_1, -9.8t + v_2) \). What are \( v_1 \) and \( v_2 \)? We find these values by decomposing the initial speed into horizontal and vertical components using trigonometry:

Thus, \( v_1 = 50 \cos 30 = 25\sqrt{3} \approx 43.301 \), \( v_2 = 50 \sin 30 = 25 \), and the velocity vector is

\[
v(t) = (43.301, -9.8t + 25).
\]

For the displacement vector, we set the origin at the base of the cliff, so that \( r_1 = 0 \) and \( r_2 = 100 \). Therefore, the displacement vector is

\[
r(t) = (43.301t, -4.9t^2 + 25t + 100).
\]
To answer the 1st question, we note that the ball reaches its maximum height when the $y$-component of the velocity vector is 0 (i.e. the ball comes to a momentary “stop” in the up-down direction). From $v(t) = (43.301, -9.8t + 25)$, we set $-9.8t + 25 = 0$ and solve for $t$:

$$ t = \frac{25}{9.8} = 2.551 \text{ seconds}. $$

This $t$ is evaluated into the $y$-component for the displacement function:

$$ \text{Maximum height} = -4.9(2.551)^2 + 25(2.551) + 100 = 131.888 \text{ m}. $$

For Question 2, the ball hits the ground when its height is 0, so we set the $y$-component of displacement to 0 and solve for $t$. If we use the quadratic formula, we’ll get a negative answer, which can be ignored. We want just the positive answer.

$$ -4.9t^2 + 25t + 100 = 0 \rightarrow t = \frac{-25 \pm \sqrt{25^2 - 4(-4.9)(100)}}{2(-4.9)} \rightarrow t = -2.637 \text{ or } 7.739. $$

The ball lands at $t = 7.739$ seconds. The distance from the base is found by evaluating the $x$-component of the displacement by this $t$:

$$ 43.301(7.739) = 335.106 \text{ m}. $$

To answer Question 3, we find the velocity vector at $t = 7.739$ seconds:

$$ v(7.739) = (43.301, -9.8(7.739) + 25) = (43.301, -50.842). $$

The speed is the magnitude of this vector:

$$ \text{Speed} = |(43.301, -50.842)| = \sqrt{43.301^2 + (-50.842)^2} \approx 66.702 \text{ m/s}. $$

Lastly, to answer Question 4, we sketch a diagram to be sure we have the components of velocity properly in place:

Looking at the picture, it makes sense to use an arctangent calculation. The angle will be acute. Thus, the angle of impact is $\theta = \tan^{-1} \frac{50.842}{43.301} = 49.6$ degrees.

In case you’re asking yourself “why don’t we just use formulas from mechanics in physics?” The answer is: this is the same thing. Physics likes formulas for each situation, meaning about 20 formulas in all. Math likes to be more general and to construct the problem from a general beginning.