Exponential Functions and Their Graphs

An exponential function is of the form \( f(x) = a^x \), where the base is a constant and the exponent is a variable. The base \( a \) must be positive and not equal to 1. Please note that these functions are not polynomials, and please note the qualitative differences between the expressions \( 2^x \) (an exponential) and \( x^2 \) (a polynomial).

If the base \( a > 1 \), then the function/graph is called exponential growth. If \( 0 < a < 1 \), then the graphs are reflected about the y-axis, and we have exponential decay. Look in your book for basic examples of growth and decay. Please note important ‘landmarks’ like the domain, range, y-intercept, the lack of x-intercepts, increasing and decreasing, etc. Be able to recognize these graphs at a glance, too.

Note: There are two ways to write exponential decay: \( f(x) = a^x \), where \( 0 < a < 1 \), or \( f(x) = a^{-x} \), where \( a > 1 \). Can you see why these are equivalent expressions? You’ll probably see both forms in this chapter but generally, we usually keep the base greater than 1, so the second form is more common. The negative in the exponent is a nice reminder we have decay.

The form \( f(x) = a^{x-h} + k \) simply includes vertical \((k)\) and horizontal \((h)\) shifts. Note that the y-intercept will change (how?), the x-intercepts may come into play (or not) and the range will differ (how?).

The natural base “\( e \)” approximately equal to 2.718, is irrational, and is commonly found in natural growth/decay models such as population, elemental decay, monetary growth, etc. Find \( e \) on your calculator and explore. Try \( e^1 \). Why \( e \)? Actually, any base is useful, but it turns out that \( e \) has some wonderfully simple properties that become relevant in calculus. Other than that, it’s just a number like any other.

Logarithmic Functions and Their Graphs

A logarithm function is the inverse of an exponential function. Logarithms are absolutely vital for solving for variables located in an exponent. Consider the difficulty of solving for \( x \) in \( 3^x = 12 \). You may be able to try some numbers at random (hint: 4 does not work), and eventually through trial and error, get close to the correct value. After we develop some basic rules, we will be able to solve this equation with no difficulty.

Fact: the words logarithm and exponent are synonymous. A logarithm is an exponent. If we consider the statement \( 3^2 = 9 \), the 2 is the exponent, of course. We can also say that the 2 is the logarithm, base 3, of 9. This would be written \( 2 = \log_3 9 \).

Practice writing and re-writing expressions in and out of logarithm form. For some practice, go here: [http://math.la.asu.edu/~surgent/logs.pdf](http://math.la.asu.edu/~surgent/logs.pdf)
You absolutely, positively need to know and memorize these three log properties:

\[
\begin{align*}
\log AB &= \log A + \log B \\
\log \frac{A}{B} &= \log A - \log B \\
\log A^n &= n \log A
\end{align*}
\]

Your text should give short proofs of these laws. Please review and practice, practice and practice!

The graph of a logarithmic function can be inferred by using the technique of reflecting across the 45-degree line the graph of an exponential function. Be sure to know the domain, range, x-intercept and other useful descriptive features of a typical log graph.

We now make note of two useful logarithms, the \textit{common logarithm} (base-10) and the \textit{natural logarithm} (base-\(e\)). Common logarithms are written \(f(x) = \log x\), where the absence of a subscript is understood to represent base-10. Natural logarithms are written \(f(x) = \ln x\). Your calculator should have these two keys. What happens if you type in a negative number and attempt to find its log or \(\ln\)? Why? The properties for the \(\ln\) are identical for those of the general log.

\textbf{Properties of Logarithms}

We now delve into the very important properties of logarithms. The first is the \textit{change-of-base} theorem. This allows us to evaluate any logarithm by converting it into a quotient of two logarithms. We typically convert into base-10 or base-\(e\) logarithms to evaluate. For example, \(\log_5 25 = \frac{\log 25}{\log 5}\), or \(\frac{\ln 5}{\ln 2}\). Enter both quotients on your calculator and verify that both versions are identical. Be sure to use parentheses properly; e.g., you’ll type in \(\log(5)/\log(2)\) or \(\ln(5)/\ln(2)\).

Again, you are reminded to commit to memory the three major properties of logs and the assortment of minor properties, all of which can be found on my practice sheet link (above).

Note: Be sure you know and understand these properties before proceeding further!

Opinion: Most students usually struggle initially with logarithms, but with a little practice, they become quite adept as using them. Logarithms follow the same rules every time. It’s just a matter of getting to know the rules.

By the way, the answer to \(3^x = 12\) is to rewrite as a logarithm:

\[x = \log_3 12 = \frac{\log 12}{\log 3} = 2.2618\ldots\]

Check by evaluating \(3^{2.2618}\), which should give an answer very near to 12.
Solving Exponential and Logarithmic equations

We start this section by introducing the inverse (or cancellation) properties of exponentials and logarithms. There are two pairs: one for general bases and one for the natural base $e$. You can prove these properties to yourself by simply re-writing each of them into standard exponent form.

Remember: logarithm and exponent are synonymous.

The properties for exponents are identical to the properties of logarithms. Please study the numerous examples given in your textbook. Notice how the “ln” cancels out the “$e$”, leaving the exponent isolated, which is what we want. Use your standard algebra skills when solving for the variable. Do not be impatient when using the ln or log to cancel the base. Use it at its appropriate time, when all other terms and coefficients are removed.

Tip: To cancel base $e$, use ln. To cancel base 10, use log. Otherwise, it does not matter which version of logarithm you use. Just don’t mix the two in the same equation.

We also see how to solve equations already involving a logarithm. It is important to see that you are “taking bases.” When you insert the $e$ to cancel the ln, you must insert it as a base.

In any logarithmic equation, you must always check your answers to ensure they don’t cause problems with the domain. Remember that the logarithm has a restricted domain.

Some equations cannot be solved algebraically. There are no methods for solving general equations that feature a combination of logs and polynomials. In such a case, use your calculator to determine a solution.

Applications of Exponential and Logarithmic Functions

Exponential functions are often used to model population growth or decay, elemental decay (chemistry), monetary growth, temperature heat loss.

A special form of the exponential family is called the logistics (or sigmoidal) curve. These curves are good for modeling growth that may have limitations on its upper bound, for example, the population of a society with limited resources.

Next, we look at logarithmic models. A common use for logarithmic modeling is for the Richter scale, which measures the power unleashed by earthquakes. Because the factors of power unleashed by one quake can be so much higher than another quake, we use a log scale to bring these numbers back down to reasonable levels. The Decibel scale for sound intensity is also logarithm-based.