The Cartesian Plane

Imagine that your city has its major boulevards trending north-south, and east-west, each one mile apart. In this setting, you can describe your location from your starting point by describing how far east or west you traveled, coupled with how far north or south you traveled. You can give a very well-defined description of your location, one which anyone else could understand, and follow. Just as we use methods like this to describe our positions, it makes sense to use something similar in mathematics. Hence, the Rectangular coordinate system, also known as the Cartesian coordinate system, named in honor of Rene Descartes, who was instrumental in the early development of analytical geometry.

The rectangular coordinate system (hereafter referred to as the coordinate system where confusion is not possible) is a horizontal number line coupled with a vertical number line, both meeting at the “0” of each line. The horizontal line is called the x-axis, and the vertical line is called the y-axis, and the point of contact of the two axes is called the origin. Points are described by a pair of numbers, the x-coordinate and the y-coordinate, describing how far from the origin the point sits. In particular, pay attention to the names of the four quadrants. We will refer often to a particular quadrant in many problems. A handy mnemonic is to imagine writing the letter “C.” You start in Quadrant I, and pass through Quadrants II, III and IV consecutively.

Two very important skills that we will use are the distance formula and the midpoint formula. The distance formula is an application of the Pythagorean Theorem for right triangles. The midpoint formula simply locates the averages of the two x-coordinates and y-coordinates, respectively.

Graphs and Graphing Utilities

Suppose you had a job that paid $7 per hour. Therefore, your pay (p) is related to the number of hours you work (h). If you work 1 hour (h = 1), you get paid $7 (p = 7). This can be treated as a point (1,7). Similarly, you can form other points, such as (2,14), (3,21), (4,28), and so forth. If you plot these points on a coordinate system, you see a very evident “picture” forming before your eyes: a line! This line is a picture of the relationship p = 7h.

An equation in two variables is, literally, a complete sentence that describes how the two variables are related to one another. In the above example, p = 7h could read as “My pay is 7 times the number of hours worked.” The word “is” translates to the = symbol; equivalently, the = symbol is the verb in the equation.

Often, you may need to test a point to determine whether it makes an equation true. If we tested (5,40) into our equation p = 7h, we would insert 5 for h, and 40 for p. Doing so, we get 40 = 7(5), which is 40 = 35, which is false. Therefore, (5,40) is not a solution to p = 7h, and it is not part of its graph. On the other hand, (5,35) works. Try it!
Note: Consult the handbook for your calculator to see how to enter equations and generate graphs. Generally, these tasks aren’t too difficult but they do require practice.

Sketching basic graphs is an excellent skill in mathematics. Please peruse the many examples in this section for tips and techniques. If in doubt, plot more points than fewer.

**Lines in the Plane**

*Linear equation* are so named because its graphs are lines. The *slope* of a line is a measure of its steepness; the formula for slope is \( m = \frac{y_2 - y_1}{x_2 - x_1} \). Slope is commonly remembered by the phrase “rise over run.” Please note the common errors that can occur when finding slope, such as transposing the numbers, getting the \( x \)’s on top, or messing up the subtraction.

Note that a line that goes “uphill” as seen from left to right will have a positive slope. Similarly, lines that go “downhill” have negative slopes. A horizontal line has a slope of 0 (why?), while a vertical line has a slope that is undefined. Recall that it is not permissible to divide by zero, as is the case in a vertical line.

The *equation of a line* is found by the *point-slope form* for the equation of a line. To use this formula the slope and a point on the line must be given, or have two points on the line given. Insert these numbers and simplify, isolating the \( y \) variable. Your answer will be of the form \( y = mx + b \), commonly known as the *slope-intercept* form for the equation of a line. This form is useful since the \( m \) value represents the slope, and the \( b \) value represents the \( y \)-intercept.

The *general form* \((ax + by = c)\) is only used in special settings. It can be determined by taking an equation and simply pacing all of the variable terms on one side of the equal sign. It has the advantage of avoiding fractional coefficients. We will use the slope-intercept form to represent lines more often.

Parallel lines have the same slope; this is intuitive. Perpendicular lines have slopes that are negative reciprocals of one another. This may not be so intuitive, but we will be able to show this fact once we have developed our skills in trigonometry later in this course. Also, note the special case of a vertical line and a horizontal line being perpendicular, even though it is not possible to consider the reciprocal of a zero-slope.

**Functions**

We now discuss the notion of a *function*. Functions are central to higher mathematics in the same way words are to writing. As such, your understanding of and your skills in manipulating functions will govern your progress accordingly. In other words, you should study this concept carefully!
A function is a relationship between two variables, where one variable is deemed the “input” variable, and the other variable deemed the “output” variable, and where one input value corresponds with one output value. This last phrase is extremely important! One input CANNOT correspond to two outputs, for example. The set of input values is known as the domain of the function, and the set of output values is the range of the function. Also, the input variable is known formally as the independent variable, while the output variable is the dependent variable.

For example, the equation $p = 7h$ is a function. The “$h$” is the independent variable, and the “$p$” is the dependent variable. Specifically, the value of $p$ is dependent on the value of $h$. Once you choose a value for $h$, the corresponding value for $p$ can thus determined.

Function notation is useful shorthand for describing functions. Instead of writing $y = 2x + 3$, we write $f(x) = 2x + 3$. The $f$ is this function’s “name,” while the $x$ reminds us what our independent variable is. The expression $f(2)$ tells us to replace $x$ with 2 in the function named $f$, getting $2(2) + 3 = 7$. Therefore, $f(2) = 7$, and we have just generated a point, the point $(2,7)$. Function notation has these advantages:

1. We can name our functions and keep confusion to a minimum. Suppose we had $f(x) = 2x + 3$ and $g(x) = 3x - 1$. Then the expression $g(4)$ is very clear! It says, “Plug in 4 for $x$ in function named $g$.” But consider the situation if we had $y = 2x + 3$ and $y = 3x - 1$. Both functions are named $y$. This can be confusing.

2. We are not limited to replacing the independent variable with a number only. We can insert anything for $x$! For example, $f(x + 2)$ means insert “$x + 2$” in for $x$ in the function $f$, getting $2(x + 2) + 3$, which equals $2x + 7$ upon simplification. We will use this idea many times.

We must be careful to not insert values into functions that might cause problems from an arithmetical standpoint. For example, if $k(x) = \frac{1}{x}$, then we can not set $x = 0$, since that would require us to divide 1 by 0, which is impossible. The domain of a function is the set of permissible input values. In this case, the domain for function $k$ would be all values of $x$ except 0.

**Graphs of Functions**

The Vertical Line Test (VLT) is an easy visual method for determining whether a graph is the graph of a function. Remember that one input value can only correspond to one output value. Hence, if we visualize vertical lines superimposed on a graph, each vertical line should intersect the graph no more than once. In such a case, the graph “passes” the VLT and is the graph of a function. If any vertical line intersects the graph more than once, the graph “fails” and it is NOT the graph of a function (in this case the graph is called a relation).
You should be familiar with the following ‘descriptive’ concepts pertaining to functions, many of which are vital in calculus. Here is a short glossary:

**Increasing:** The graph goes “up” as one reads from left to right.

**Decreasing:** The graph goes “down” as one reads from left to right.

**Constant:** The graph is horizontal.

**Minimum/Minima:** Points on a graph that are lower than the immediate surrounding graph.

**Maximum/Maxima:** Points on a graph that are higher than the immediate surrounding graph.

**Relative:** the point is lower/higher than its immediate neighbors.

**Absolute:** The point is lowest/highest of the entire graph.

These ideas are used all the time in real life. For example, we “maximize profits” while “minimizing costs”; “Inflation decreased for many years, before minimizing (bottoming out), and then increasing slightly.” Can you think of situations where you have used these terms?

**Even:** The function is symmetrical across the y-axis (e.g. $x^2, |x|$).

**Odd:** The function is symmetrical across the origin (e.g. $x^3$)

Later we shall see that the sine function is odd, the cosine function is even.

**Piecewise:** a function composed of bits and pieces of other functions.

**Discrete:** A function composed of individual points.