Chapter 7: Trigonometry

Trigonometry is the study of angles and how they can be used as a means of indirect measurement, that is, the measurement of a distance where it is not practical or even possible to measure it directly. For example, surveyors use trigonometry to measure the heights of mountains and distances across bodies of water, while engineers and architects use trigonometry to ensure their buildings are being built to exact specifications. Also, any phenomena that repeats itself on a periodic basis—the seasons, fluctuations in sales, heartbeats—can be modeled by trigonometric functions.

Section 7.1: Introduction to Trigonometry

Angles and Quadrants

We start with a discussion of angles.

A ray is placed with its endpoint at the origin of an $xy$-axis system, with the ray itself lying along the positive $x$-axis. The ray is allowed to rotate. Counterclockwise rotations are considered positive, and clockwise rotations are called negative (See Fig 1).

The $xy$-axis system itself is subdivided into four quadrants, each defined by the $x$- and $y$-axes. These quadrants are numbered 1 through 4, with Quadrant 1 being at the top right (where both $x$ and $y$ are positive) and Quadrants 2, 3 and 4 following consecutively as the ray makes one complete rotation around the plane (See Fig 2).

The words angle and rotation are synonymous with one another. An angle is measured from the ray’s starting position along the positive $x$-axis (called its initial side) and ending at its terminal side.

Angle Relationships and Degree Measurement

Degree measurement is based on a circle, which is 360 degrees, or $360^\circ$. If a ray is allowed to rotate one complete revolution counterclockwise around the $xy$-plane, starting and ending at the positive $x$-axis, we say that the ray has rotated 360 degrees. Its angle measurement would be written $360^\circ$.

Smaller (or larger) rotations can then be defined proportionally. For example, if the ray rotates half-way around the plane in the counterclockwise direction, from the positive $x$-axis to the negative $x$-axis, we say that the ray rotated 180 degrees (half of 360 degrees), and that its angle measurement is $180^\circ$. Similarly, if the ray rotates from the positive $x$-axis to the positive $y$-axis in a counterclockwise direction, its rotation is 90 degrees and its angle measurement is $90^\circ$. 
Example 1: State the angle measurement in degrees for the following angles.

1. Initial side: positive x-axis, terminal side: negative y-axis, counter-clockwise rotation.
2. Initial side: positive y-axis, terminal side: positive x-axis, clockwise rotation.
3. Initial side: negative x-axis, terminal side: negative y-axis, counter-clockwise rotation.
4. Initial side: positive y-axis, terminal side: negative x-axis, clockwise rotation

Solution:

1. 270°
2. –90°
3. 90°
4. –270°

Some angles have special names: an angle of 180° is called a straight angle, and an angle of 90° is called a right angle. Angles between 90° and 180° are called obtuse and angles between 0° and 90° are called acute. When two angles sum to 180°, they are called supplementary. When two angles sum to 90°, they are called complementary. (See Fig 3).

When two angles have the same initial and terminal sides, they are called coterminal. In fact, every angle has an infinite number of equivalent coterminal angles, since we may allow the ray to rotate many times about the plane, in either the positive or negative directions. For example, if a ray rotates counterclockwise from the positive x-axis to the positive y-axis, it forms an angle of 90°. Equivalently, in the clockwise direction, this angle has a measure of –270°. These two angles are coterminal; see Fig. 4:

If the ray was allowed to rotate more than once before ending at its terminal side, other coterminal angle measures would be 450°, 810° and 1170° (one, two and three complete extra revolutions in the positive
direction), and \(-630^\circ, -990^\circ\), and \(-1350^\circ\) (one, two and three complete extra revolutions in the negative direction). In practice, we usually choose the smallest measure in most applications (See Fig 5).

\[\begin{align*}
\text{Fig 5}
\end{align*}\]

**Example 2:** Assume two rays meet, forming an angle of \(27^\circ\). State this angle’s (a) supplement, (b) complement, (c) coterminal in the opposite direction, and (d) coterminal with one extra revolution in the positive direction.

**Solution:**

a) Its supplement is \(180^\circ - 27^\circ = 153^\circ\).

b) Its complement is \(90^\circ - 27^\circ = 63^\circ\).

c) Its coterminal measure would be \(360^\circ - 27^\circ = 323^\circ\), but since it is traced in the opposite direction, it would have a negative value. Therefore, its coterminal measure in the opposite direction is \(-323^\circ\).

d) Its coterminal measure with one extra revolution in the positive direction is \(27^\circ + 360^\circ = 387^\circ\).

**Example 3:** A screw is turned 6 complete rotations, plus another half rotation, before it is set tight. What would be the angle through which this screw was turned?

**Solution:** Six complete rotations would give an angle of \(6 \times 360^\circ = 2,160^\circ\), and the extra half-rotation would be an extra \(180^\circ\). Thus, the screw was turned a total of \(2,340^\circ\).

**Example 4:** Determine the following angles:

1. Two angles are supplementary, with the second angle being twice as large as the first angle.
2. Two angles are complementary, with the second angle being one less than three times the smaller angle.

**Solution:**

1. Let \(\theta\) be the smaller angle’s measure. Therefore, the larger angle has a measure of \(2\theta\). Since they are supplementary, their sum is \(180^\circ\). We solve for \(\theta\):

\begin{align*}
\theta + 2\theta &= 180 \\
3\theta &= 180 \\
\theta &= 60
\end{align*}

The smaller angle is \(60^\circ\) and the larger angle is \(120^\circ\).
2. Letting $\theta$ be the smaller angle, the larger angle is $3\theta - 1$, and their sum is $90^\circ$. We solve for $\theta$:

\[
\begin{align*}
\theta + (3\theta - 1) &= 90 \\
4\theta - 1 &= 90 \\
4\theta &= 91 \\
\theta &= \frac{91}{4} = 22.75
\end{align*}
\]

The smaller angle is $22.75^\circ$ and the larger angle is $67.25^\circ$.

Comment: the Greek letter $\theta$ (theta) is often used to represent an angle.

**Radian Measurement**

Suppose we overlay a circle with radius 1 on the $xy$-plane, its center at the origin. This is called the *unit circle*, and its circumference is $C = 2\pi(1) = 2\pi$. A ray is drawn from the origin, intersecting the circle at point $P$ (see Fig. 6). This ray is the terminal side of an angle, with the initial side being the positive $x$-axis, and this angle “cuts off” (subtends) a portion of the circle. The portion of the circle from $(1,0)$ to point $P$ (in the positive direction) is called an *arc* of the unit circle, and its length will be proportional to the circle’s circumference (which is $2\pi$) in the same way the angle’s measure will be to $360^\circ$:

![Fig 6](image)

\[
\frac{\text{length of arc (radian measure)}}{2\pi} = \frac{\text{angle in degrees}}{360^\circ}
\]

The length of the arc is called the *radian measure* of the subtending angle.

A complete unit circle covers $360^\circ$ and has a circumference of $2\pi$. Therefore, we can relate these two facts as follows:

\[
360 \text{ degrees} = 2\pi \text{ radians}
\]

Dividing both sides by $360$ gives us

\[
1 \text{ degree} = \frac{\pi}{180} \text{ radians} \quad \frac{2\pi}{360} = \frac{\pi}{180}
\]

Or, we can divide by $2\pi$, which gives us

\[
\frac{180}{\pi} \text{ degrees} = 1 \text{ radian}. \quad \frac{360}{2\pi} = \frac{180}{\pi}
\]
Therefore, we have two conversion formulas:

1. To convert from degrees to radians, multiply the degree measure by \( \frac{\pi}{180} \).
2. To convert from radians to degrees, multiply the radian measure by \( \frac{180}{\pi} \).

Note: radian measure does not use the degree symbol °, and often, the fraction is reduced but the π is left alone.

**Example 5:** Perform the following conversions.

1. Convert 100° into radians.
2. Convert \( \frac{\pi}{5} \) radians into degrees.

**Solution:**

1. We multiply 100 by \( \frac{\pi}{180} \) and reduce:

   \[
   100 \left( \frac{\pi}{180} \right) = \frac{5}{9} \pi.
   \]

   Therefore, 100° is equivalent to \( \frac{5}{9} \pi \) radians.

2. We multiply \( \frac{\pi}{5} \) by \( \frac{180}{\pi} \) and reduce:

   \[
   \frac{\pi}{5} \cdot \frac{180}{\pi} = \frac{180}{5} = 32^\circ.
   \]

   Notice that \( \pi/\pi = 1 \). Therefore, \( \frac{\pi}{5} \) radians is equivalent to 32°.

Some common angles can be quickly determined. These are very common and should be committed to memory:

- Half circle (straight angle): \( 180^\circ = \pi \) radians.
- Quarter circle (right angle): \( 90^\circ = \frac{\pi}{2} \) radians.
- Eighth circle: \( 45^\circ = \frac{\pi}{4} \) radians.

The four quadrants can be defined in both degree and radian measurement. Let \( \theta \) represent the value of the angle.

<table>
<thead>
<tr>
<th>Quadrant</th>
<th>Degree Range</th>
<th>Radian Range</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0 &lt; ( \theta ) &lt; 90</td>
<td>0 &lt; ( \theta ) &lt; ( \frac{\pi}{2} )</td>
</tr>
<tr>
<td>2</td>
<td>90 &lt; ( \theta ) &lt; 180</td>
<td>( \frac{\pi}{2} &lt; \theta &lt; \pi )</td>
</tr>
<tr>
<td>3</td>
<td>180 &lt; ( \theta ) &lt; 270</td>
<td>( \pi &lt; \theta &lt; \frac{3\pi}{2} )</td>
</tr>
<tr>
<td>4</td>
<td>270 &lt; ( \theta ) &lt; 360</td>
<td>( \frac{3\pi}{2} &lt; \theta &lt; 2\pi )</td>
</tr>
</tbody>
</table>
Note the strict inequalities. Angles that are exactly $0^\circ, 90^\circ, 180^\circ, 270^\circ$ or $360^\circ$ (in radians: $0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}$ or $2\pi$) are called *cardinal* angles. They are not part of any quadrant.

**The Trigonometric Functions**

Let us look again at a unit circle (radius = 1) placed on an $xy$-axis system with its center at the origin. A ray is drawn, intersecting the circle at point $P$ as shown in Fig. 7. Let $\theta$ represent the angle from the positive $x$-axis to the ray.

Three definitions can now be made, each corresponding to a physical characteristic of the ray and its intersection of the unit circle at point $P$:

<table>
<thead>
<tr>
<th>DEFINITION:</th>
</tr>
</thead>
<tbody>
<tr>
<td>• The <strong>cosine</strong> of $\theta$, written $\cos \theta$, is the $x$-coordinate of $P$.</td>
</tr>
<tr>
<td>• The <strong>sine</strong> of $\theta$, written $\sin \theta$, is the $y$-coordinate of $P$.</td>
</tr>
<tr>
<td>• The <strong>tangent</strong> of $\theta$, written $\tan \theta$, is the slope of the ray.</td>
</tr>
</tbody>
</table>

![Fig. 7](image)

With these definitions, some immediate results can be determined for the cardinal angles based on the unit circle:

- For $0^\circ$ ($0$ radians), this is the point $(1,0)$. Therefore, $\cos(0) = 1$, $\sin(0) = 0$ and the slope of the ray is $\tan(0) = 0$.
- For $90^\circ$ ($\pi/2$ radians), this is the point $(0,1)$. Therefore, $\cos(90) = 0$, $\sin(90) = 1$ and since the ray is vertical, the slope is not defined (that is, $\tan(90)$ is not defined).
- For $180^\circ$ ($\pi$ radians), this is the point $(-1,0)$. Therefore, $\cos(180) = -1$, $\sin(180) = 0$ and the slope of the ray is $\tan(180) = 0$.
- For $270^\circ$ ($3\pi/2$ radians), this is the point $(0,-1)$. Therefore, $\cos(270) = 0$, $\sin(270) = -1$ and the slope of the ray is undefined since it is vertical.
For most other angles, a calculator is necessary to calculate the cosine, sine and tangent of an angle.

**Example 6:** A ray with an angle of $50^\circ$ is drawn, with initial side being the positive $x$-axis. State the point at which this ray intersects the circle, and the slope of this ray.

**Solution:** Using a calculator set to “Degree” mode, the $x$-coordinate is $\cos(50) = 0.643$ (rounded), and the $y$-coordinate is $\sin(50) = 0.766$ (rounded). The slope of the ray is $\tan(50) = 1.192$ (rounded).

We can also observe some geometrical relationships between sine, cosine and tangent. First, we can view the portion of the ray from the origin to the circle as a hypotenuse (of length 1) of a right triangle. The legs have length $\cos \theta$ (called the *adjacent leg*) and $\sin \theta$ (called the *opposite leg*) (See Fig. 8).

Therefore, we can use the Pythagorean formula to form a relationship between cosine and sine:

$$(\text{adjacent leg})^2 + (\text{opposite leg})^2 = (\text{hypotenuse})^2$$

$$(\cos \theta)^2 + (\sin \theta)^2 = 1^2$$

This is usually written as

$$\cos^2 \theta + \sin^2 \theta = 1.$$
This identity is called the Pythagorean Identity. Notice that we write $\cos^2 \theta$ as a short-hand for $(\cos \theta)^2$.

Another very common relationship between sine, cosine and tangent uses the slope formula. Since the ray passes through the origin $(0,0)$ and point $P$, which is $(\cos \theta, \sin \theta)$, we can calculate the slope of the ray, which is defined to be $\tan \theta$:

$$\text{Slope of ray} = \tan \theta = \frac{\sin \theta - 0}{\cos \theta - 0}$$

Therefore, we have

$$\tan \theta = \frac{\sin \theta}{\cos \theta}$$

**Trigonometric Values of Special Angles**

Exact values for the sine, cosine and tangent of some special angles can be determined using geometry. We look at the case of $45^\circ \left(\frac{\pi}{4} \text{ radians}\right)$ first:

Suppose the ray is drawn with an angle of $45^\circ$ on the unit circle as seen in Figure 9 below. Let $P$ be its intersection with the circle.

![Fig. 9](image)

The hypotenuse has length 1, and its $x$ and $y$ coordinates will be the same. Using the Pythagorean formula, we have

$$x^2 + x^2 = 1 \quad \text{Remember, } x = y.$$  
$$2x^2 = 1$$  
$$x^2 = \frac{1}{2}$$  
$$x = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}. \quad \text{Rationalizing the denominator.}$$

The $x$-coordinate of $P$ is $\frac{\sqrt{2}}{2}$, and the $y$-coordinate of $P$ is also $\frac{\sqrt{2}}{2}$. Therefore,

$$\cos(45) = \frac{\sqrt{2}}{2} \text{ and } \sin(45) = \frac{\sqrt{2}}{2}.$$  

Since the slope is 1, we observe that $\tan(45) = 1$. 
Using radians, we have \( \cos \left( \frac{\pi}{4} \right) = \frac{\sqrt{2}}{2} \), \( \sin \left( \frac{\pi}{4} \right) = \frac{\sqrt{2}}{2} \) and \( \tan \left( \frac{\pi}{4} \right) = 1 \).

Now suppose we draw the ray at an angle of \( 60^\circ \left( \frac{\pi}{3} \text{ radians} \right) \), intersecting the unit circle at point \( P \). Recall from geometry that an equilateral triangle (all sides equal in length) has internal angles of \( 60^\circ \). Therefore, we draw a line from \( P \) to the point \((1,0)\) and form an equilateral triangle with sides of length 1. Now divide this triangle in half by sketching a vertical line from \( P \) to the point \((\frac{1}{2}, 0)\) (See Fig 10).

![Figure 10](image)

We now have a right triangle with hypotenuse of length 1 and an adjacent leg of length \( \frac{1}{2} \). Using the Pythagorean formula, the opposite leg has length \( \frac{\sqrt{3}}{2} \). Point \( P \)'s coordinate is \( \left( \frac{1}{2}, \frac{\sqrt{3}}{2} \right) \) and we have

\[
\cos(60) = \frac{1}{2} \quad \text{and} \quad \sin(60) = \frac{\sqrt{3}}{2}.
\]

The slope of this ray is

\[
\tan(60) = \frac{\sin(60)}{\cos(60)} = \frac{\frac{\sqrt{3}}{2}}{\frac{1}{2}} = \sqrt{3}.
\]

In radians, we have \( \cos \left( \frac{\pi}{3} \right) = \frac{1}{2}, \sin \left( \frac{\pi}{3} \right) = \frac{\sqrt{3}}{2} \) and \( \tan \left( \frac{\pi}{3} \right) = \sqrt{3} \).

A similar construction shows that \( \cos(30) = \frac{\sqrt{3}}{2}, \sin(30) = \frac{1}{2} \) and \( \tan(30) = \frac{\sqrt{3}}{3} \). In radians, we have \( \cos \left( \frac{\pi}{6} \right) = \frac{\sqrt{3}}{2}, \sin \left( \frac{\pi}{6} \right) = \frac{1}{2} \) and \( \tan \left( \frac{\pi}{6} \right) = \frac{\sqrt{3}}{3} \). This is left as homework exercise XX.

These values are summarized in the following table:

<table>
<thead>
<tr>
<th>Angle ( \theta ) (Degrees)</th>
<th>Angle ( \theta ) (Radians)</th>
<th>( \cos \theta )</th>
<th>( \sin \theta )</th>
<th>( \tan \theta )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>( 30^\circ )</td>
<td>( \pi/6 )</td>
<td>( \sqrt{3}/2 )</td>
<td>( 1/2 )</td>
<td>( \sqrt{3}/3 )</td>
</tr>
<tr>
<td>( 45^\circ )</td>
<td>( \pi/4 )</td>
<td>( \sqrt{2}/2 )</td>
<td>( \sqrt{2}/2 )</td>
<td>1</td>
</tr>
<tr>
<td>( 60^\circ )</td>
<td>( \pi/3 )</td>
<td>( 1/2 )</td>
<td>( \sqrt{3}/2 )</td>
<td>( \sqrt{3} )</td>
</tr>
<tr>
<td>( 90^\circ )</td>
<td>( \pi/2 )</td>
<td>0</td>
<td>1</td>
<td>undefined</td>
</tr>
</tbody>
</table>

Note that these values are for the first quadrant.
In Quadrant 2, the cosine will be negative (since the x-coordinate of any point in Quadrant 2 is negative). The sine will stay positive, and since rays in Quadrant 2 have negative slope, the tangent will be negative.

In Quadrant 3, both the cosine and sine values are negative, but rays in this quadrant have positive slope, so the tangent is positive.

In Quadrant 4, the cosine is positive, the sine is negative and the tangent is negative.

This is summarized in the following figure:

<table>
<thead>
<tr>
<th>Quadrant 2</th>
<th>Quadrant 1</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \cos \theta &lt; 0 )</td>
<td>( \cos \theta &gt; 0 )</td>
</tr>
<tr>
<td>( \sin \theta &gt; 0 )</td>
<td>( \sin \theta = 0 )</td>
</tr>
<tr>
<td>( \tan \theta &lt; 0 )</td>
<td>( \tan \theta &gt; 0 )</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Quadrant 3</th>
<th>Quadrant 4</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \cos \theta &lt; 0 )</td>
<td>( \cos \theta &gt; 0 )</td>
</tr>
<tr>
<td>( \sin \theta &lt; 0 )</td>
<td>( \sin \theta &lt; 0 )</td>
</tr>
<tr>
<td>( \tan \theta &gt; 0 )</td>
<td>( \tan \theta &lt; 0 )</td>
</tr>
</tbody>
</table>

Remember, “All Students Take Calculus”. In Quadrant 1, all trigonometric functions are positive. In Quadrant 2, the sine function is positive. In Quadrant 3, the tangent function is positive, and in Quadrant 4, the cosine function is positive.

We use symmetry to determine the actual values for cosine, sine and tangent.

**Example:** Suppose \( \theta = 120^\circ \). Determine \( \cos \theta \), \( \sin \theta \) and \( \tan \theta \).

**Solution:** Since \( 120^\circ \) is in Quadrant 2, the cosine will be negative, the sine positive and the tangent negative. The ray that represents \( 120^\circ \) is drawn, and we notice that it is symmetric across the y-axis with the ray for \( 60^\circ \). We know from the above table that \( \cos 60 = \frac{1}{2} \), \( \sin 60 = \frac{\sqrt{3}}{2} \) and \( \tan 60 = \sqrt{3} \). We attach a negative sign to the cosine and tangent values, and leave the sine value alone. Therefore, \( \cos 120 = -\frac{1}{2} \), \( \sin 120 = \frac{\sqrt{3}}{2} \) and \( \tan 120 = -\sqrt{3} \).

In the above example, the \( 60^\circ \) was the reference angle used to determine the trigonometric values for \( 120^\circ \). The reference angle lies in Quadrant 1, and to determine the correct reference angle for an angle \( \theta \) in Quadrants 2, 3 or 4, refer to the table below:

<table>
<thead>
<tr>
<th>Quadrant</th>
<th>Reference Angle Formula</th>
<th>Symmetry</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>( 180 - \theta ) (degrees) ( \pi - \theta ) (radians)</td>
<td>Across y-axis</td>
</tr>
<tr>
<td>3</td>
<td>( \theta - 180 ) (degrees) ( \theta - \pi ) (radians)</td>
<td>Across origin</td>
</tr>
<tr>
<td>4</td>
<td>( 360 - \theta ) (degrees) ( 2\pi - \theta ) (radians)</td>
<td>Across x-axis</td>
</tr>
</tbody>
</table>
**Graphs of the Trigonometric Functions**

If we treat $\theta$ as a variable representing the angle of a ray as it rotates counterclockwise in the $xy$-plane, we may then treat the cosine, sine and tangents as functions of $\theta$:

\[
\begin{align*}
    f(\theta) &= \cos \theta \\
    g(\theta) &= \sin \theta \\
    h(\theta) &= \tan \theta
\end{align*}
\]

Note: when graphing any trigonometric function, the angle (input) $\theta$ is always in radians.

- Graph of the function $f(\theta) = \cos \theta$.

We can create a table of values for this function, as $\theta$ makes one complete positive rotation about the $xy$-plane.

<table>
<thead>
<tr>
<th>$\cos(0)$</th>
<th>$\cos(\pi/2)$</th>
<th>$\cos(\pi)$</th>
<th>$\cos(3\pi/2)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$1$</td>
<td>$0$</td>
<td>$-1$</td>
<td>$0$</td>
</tr>
<tr>
<td>$\cos(\pi/6)$ = $\sqrt{3}/2$</td>
<td>$\cos(2\pi/3)$ = $-1/2$</td>
<td>$\cos(7\pi/6)$ = $-\sqrt{3}/2$</td>
<td>$\cos(5\pi/3)$ = $1/2$</td>
</tr>
<tr>
<td>$\cos(\pi/4)$ = $\sqrt{2}/2$</td>
<td>$\cos(3\pi/4)$ = $-\sqrt{2}/2$</td>
<td>$\cos(5\pi/4)$ = $-\sqrt{2}/2$</td>
<td>$\cos(7\pi/4)$ = $\sqrt{2}/2$</td>
</tr>
<tr>
<td>$\cos(\pi/3)$ = $1/2$</td>
<td>$\cos(5\pi/6)$ = $-\sqrt{3}/2$</td>
<td>$\cos(4\pi/3)$ = $-1/2$</td>
<td>$\cos(11\pi/6)$ = $\sqrt{3}/2$</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>$\cos(2\pi)$ = $1$</td>
</tr>
</tbody>
</table>

Plotting these points and connecting them with a smooth curve generates a portion of the cosine graph:

The complete graph of $f(\theta) = \cos \theta$ is defined for $-\infty < \theta < \infty$. Notice that the “wave” repeats itself every $2\pi$ units. The range of $f(\theta) = \cos \theta$ is $-1 \leq \cos \theta \leq 1$.

- Graph of the function $g(\theta) = \sin \theta$

In a similar manner, the graph of the function $g(\theta) = \sin \theta$ is generated
It also repeats itself every $2\pi$ units, and it too has a range of $-1 \leq \sin \theta \leq 1$.

- The graph of the function $h(\theta) = \tan \theta$

The tangent function is undefined when $\theta = \pm \frac{\pi}{2}, \pm \frac{3\pi}{2}, \pm \frac{5\pi}{2}, \ldots$. These angles will force the ray to be vertical, in which case the slope of the ray is undefined. On the graph, there will be vertical asymptotes at these values.

Other values can be generated using methods we have discussed previously. The complete graph of the tangent function is:

![Tangent Function Graph](image)

The range of the tangent function is $-\infty < \tan \theta < \infty$. The portion of the tangent function between $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$ is called the *main branch* of the tangent function. These branches repeat forever, with a period of $\pi$ units.
**Periodicity**

Since the ray can rotate infinitely often, the function (output) values will repeat in a consistent pattern. In the case of the cosine and sine functions, these values will repeat every time the ray has completed a full rotation, that is, it has swept out an angle of $2\pi$ radians. Therefore, we have

\[
\cos(\theta + 2\pi) = \cos \theta \\
\sin(\theta + 2\pi) = \sin \theta
\]

This leads us to a definition of periodicity.

**DEFINITION**

A function is **periodic** if there exists a value $p$ such that $f(x + p) = f(x)$, for all $x$ in the domain of $f$. The smallest such value of $p$ is called the **period** of the function.

Therefore, the cosine and sine function each have period $2\pi$ (If the angle is given in degrees, the period is $360^\circ$). The tangent function has a period of $\pi$ units. That is, \( \tan(\theta + \pi) = \tan \theta \). These facts can be used to “reduce” large angles into smaller equivalents, as the next example shows:

**Example:** Determine the following measurements:

1. $\sin\left(\frac{7\pi}{3}\right)$
2. $\cos\left(\frac{21\pi}{4}\right)$
3. $\tan\left(\frac{7\pi}{6}\right)$

**Solution:**

1. The angle $\frac{7\pi}{3}$ radians can be reduced by recognizing that it is $\frac{\pi}{3} + 2\pi$. Therefore,

   \[
   \sin\left(\frac{7\pi}{3}\right) = \sin\left(\frac{\pi}{3} + 2\pi\right) = \sin\left(\frac{\pi}{3}\right).
   \]

   We know that $\sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}$. Therefore, $\sin\left(\frac{7\pi}{3}\right) = \frac{\sqrt{3}}{2}$.

2. The angle $\frac{21\pi}{4}$ is the same as $\frac{5\pi}{4} + 4\pi$. The $4\pi$ represents two periods, or two complete rotations of the angle through the $xy$-plane. We note that what remains, the $\frac{5\pi}{4}$, is in Quadrant 3, and that the reference angle is $\frac{\pi}{4}$. We know that $\cos\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$, and since the cosine is negative in Quadrant 3, we conclude that $\cos\left(\frac{21\pi}{4}\right) = -\frac{\sqrt{2}}{2}$.

3. The angle $\frac{7\pi}{6}$ is the same as $\frac{\pi}{6} + \pi$. Therefore,

   \[
   \tan\left(\frac{7\pi}{6}\right) = \tan\left(\frac{\pi}{6} + \pi\right) = \tan\left(\frac{\pi}{6}\right).
   \]
Since \( \tan\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{3} \), we conclude that \( \tan\left(\frac{7\pi}{6}\right) = \frac{\sqrt{3}}{3} \).

### Right-Triangle Trigonometry and the Reciprocal Functions

The sine, cosine and tangent functions can be defined on a right triangle. Let angle \( \theta \) be placed as shown in Fig. 11.

![Fig. 11](image)

Remember, the leg closest to this angle is called the adjacent leg, and the other leg is called the opposite leg. Therefore, the sine, cosine and tangent functions can be defined in terms of the lengths of the adjacent leg, the opposite leg and the hypotenuse:

\[
\sin \theta = \frac{\text{opposite}}{\text{hypotenuse}}, \quad \cos \theta = \frac{\text{adjacent}}{\text{hypotenuse}}, \quad \tan \theta = \frac{\text{opposite}}{\text{adjacent}}.
\]

Using “O” for opposite, “A” for adjacent and “H” for hypotenuse, the mnemonic “SOHCAHTOA” is a useful way to commit these relationships to memory. Often, the trigonometric values can be determined even if the angle is unknown, as the following example shows.

**Example:** A right triangle as an adjacent leg of length 7 and a hypotenuse of length 10. Determine exact values for \( \sin \theta \), \( \cos \theta \) and \( \tan \theta \).

**Solution:** We use the Pythagorean Formula to determine the length of the opposite leg (denoted \( y \)):

\[
7^2 + y^2 = 10^2 \\
49 + y^2 = 100 \\
y^2 = 51 \\
y = \sqrt{51}
\]

Therefore,

\[
\sin \theta = \frac{\text{opp}}{\text{hyp}} = \frac{\sqrt{51}}{10}, \quad \cos \theta = \frac{\text{adj}}{\text{hyp}} = \frac{7}{10} \quad \text{and} \quad \tan \theta = \frac{\text{opp}}{\text{adj}} = \frac{\sqrt{51}}{7}.
\]

The reciprocal functions are defined as follows:
DEFINITION

- The secant of $\theta$, written sec $\theta$, is the reciprocal of cos $\theta$. That is, $\sec \theta = \frac{1}{\cos \theta} = \frac{\text{hypotenuse}}{\text{adjacent}}$. 
- The cosecant of $\theta$, written csc $\theta$, is the reciprocal of sin $\theta$. That is, $\csc \theta = \frac{1}{\sin \theta} = \frac{\text{hypotenuse}}{\text{opposite}}$. 
- The cotangent of $\theta$, written cot $\theta$, is the reciprocal of tan $\theta$. That is, $\cot \theta = \frac{1}{\tan \theta} = \frac{\cos \theta}{\sin \theta} = \frac{\text{adjacent}}{\text{opposite}}$.

The physical interpretations (as lengths) of all six trigonometric functions can be seen on one diagram, as shown below.

Example: Suppose $\sin \theta = \frac{4}{7}$ and that $\theta$ is in Quadrant 1. Determine the remaining five trigonometric function values of $\theta$ in exact (non-decimal) form.

Solution: In Quadrant 1, all six trigonometric functions are positive. Since $\sin \theta = \frac{4}{7}$, we know the opposite leg has length 4 and hypotenuse has length 7. The Pythagorean Formula is used to determine the length of the remaining (adjacent) leg:

$$(\text{adj})^2 + 4^2 = 7^2$$

$$(\text{adj})^2 + 16 = 49$$

$$(\text{adj})^2 = 33$$

$$\text{adj} = \sqrt{33}$$

Since $\cos \theta = \frac{\text{adjacent}}{\text{hypotenuse}}$, we conclude that $\cos \theta = \frac{\sqrt{33}}{7}$. Also, since $\tan \theta = \frac{\text{opposite}}{\text{adjacent}}$, we conclude that $\tan \theta = \frac{4}{\sqrt{33}} = \frac{4\sqrt{33}}{33}$, where the denominator was rationalized in the last step. The remaining trigonometric function values are determined by taking appropriate reciprocals:

$$\csc \theta = \frac{1}{\sin \theta} = \frac{1}{4/7} = \frac{7}{4}$$

$$\sec \theta = \frac{1}{\cos \theta} = \frac{1}{\sqrt{33}/7} = \frac{7\sqrt{33}}{33}$$

Rationalize the denominator.
\[ \cot \theta = \frac{1}{\tan \theta} = \frac{1}{\frac{\sin \theta}{\cos \theta}} = \frac{\cos \theta}{\sin \theta} = \frac{\sqrt{33}}{4}. \] Rationalize the denominator

Had the angle \( \theta \) been in a different quadrant, negative signs would have been assigned according to the “ASTC” mnemonic. For example, if \( \theta \) was in quadrant 2 in the above example, the cosine and tangent results would have been negative, and their reciprocal (the secant and cotangent) results would have been negative as well. The cosecant result would have remained positive.

For most practical measurement purposes, the cosine, sine and tangent functions are almost always sufficient. However, in calculus, the reciprocal functions (in particular, the secant function) have many uses.

**Common Identities**

There are many useful identities that we use in trigonometry and in calculus. The most common involving the cosine and sine functions are stated below.

- **The Sum and Difference Identities**
  \[
  \sin(\alpha \pm \beta) = \sin \alpha \cos \beta \pm \cos \alpha \sin \beta \\
  \cos(\alpha \pm \beta) = \cos \alpha \cos \beta \mp \sin \alpha \sin \beta 
  \]

  Note the reversal of the signs in the cosine sum-difference formula.

**Example:** Use an appropriate sum-difference formula to determine the exact value of \( \sin(75^\circ) \).

**Solution:** We see that \( 75^\circ = 45^\circ + 30^\circ \). Therefore,

\[
\sin(75^\circ) = \sin(45^\circ + 30^\circ) \\
= \sin(45^\circ) \cos(30^\circ) + \cos(45^\circ) \sin(30^\circ) \\
= \left( \frac{\sqrt{2}}{2} \right) \left( \frac{\sqrt{3}}{2} \right) + \left( \frac{\sqrt{2}}{2} \right) \left( \frac{1}{2} \right) \\
= \frac{\sqrt{6} + \sqrt{2}}{4} \quad \text{Referring to the table of 1st Quadrant values}
\]

These sum-difference identities can be used to create the double-angle identities:

- **The Double Angle Identities**
  \[
  \sin(2\theta) = 2 \sin \theta \cos \theta \\
  \cos(2\theta) = \cos^2 \theta - \sin^2 \theta 
  \]

The proofs of both are left as exercises XX and YY in the homework.

**Example:** If \( \cos \theta = \frac{1}{3} \) and \( \theta \) is in Quadrant 1, determine exact values for \( \sin(2\theta) \) and \( \cos(2\theta) \).

**Solution:** We’ll need the exact value for \( \sin \theta \), which we can get from the Pythagorean Identity—it is \( \sin \theta = \frac{2\sqrt{2}}{3} \) (You should verify this).
Therefore,
\[
\sin(2\theta) = 2 \sin \theta \cos \theta \\
= 2 \left( \frac{2\sqrt{2}}{3} \right) \left( \frac{1}{3} \right) \\
= \frac{4\sqrt{2}}{9}
\]
Also, we have
\[
\cos(2\theta) = \cos^2 \theta - \sin^2 \theta \\
= \left( \frac{1}{3} \right)^2 - \left( \frac{2\sqrt{2}}{3} \right)^2 \\
= \frac{1}{9} - \frac{8}{9} = -\frac{7}{9}
\]
Why is \(\cos(2\theta)\) negative? In this case, since \(\theta\) was in the 1st Quadrant, doubling its size evidently placed the angle \(2\theta\) into Quadrant 2, where the cosine is negative and the sine remains positive.

The Pythagorean Identity is
\[
\cos^2 \theta + \sin^2 \theta = 1.
\]
Re-arranging or dividing through by terms produces many corollary forms of the Pythagorean Identity.

- **The Pythagorean Identities and Corollaries**

\[
\begin{align*}
\cos^2 \theta + \sin^2 \theta &= 1 \\
\cos^2 \theta &= 1 - \sin^2 \theta \quad \text{(subtraction of \(\sin^2 \theta\))} \\
\sin^2 \theta &= 1 - \cos^2 \theta \quad \text{(subtraction of \(\cos^2 \theta\))} \\
1 + \tan^2 \theta &= \sec^2 \theta \quad \text{(divide through by \(\cos^2 \theta\))} \\
\tan^2 \theta &= \sec^2 \theta - 1 \quad \text{(subtraction of 1)} \\
\cot^2 \theta + 1 &= \csc^2 \theta \quad \text{(divide through by \(\sin^2 \theta\))} \\
\cot^2 \theta &= \csc^2 \theta - 1 \quad \text{(subtraction of 1)}
\end{align*}
\]

**Example:** The double angle identity for the cosine function is
\[
\cos(2\theta) = \cos^2 \theta - \sin^2 \theta.
\]
Show that this formula can also be written as
\[
\begin{align*}
1) \quad \cos(2\theta) &= 1 - 2\sin^2 \theta \\
2) \quad \cos(2\theta) &= 2\cos^2 \theta - 1
\end{align*}
\]

**Solution:** In the first case (1), we substitute \(\cos^2 \theta\) with \(1 - \sin^2 \theta\) and simplify:
\[
\cos(2\theta) = \cos^2 \theta - \sin^2 \theta \\
= (1 - \sin^2 \theta) - \sin^2 \theta \\
= 1 - 2\sin^2 \theta
\]

In case (2), we substitute \( \sin^2 \theta \) with \( 1 - \cos^2 \theta \) and simplify:

\[
\cos(2\theta) = \cos^2 \theta - \sin^2 \theta \\
= \cos^2 \theta - (1 - \cos^2 \theta) \\
= 2\cos^2 \theta - 1
\]

Homework Set

State the angle in degrees.

1. Initial side: positive \( x \)-axis, terminal side: negative \( x \)-axis, clockwise rotation.
2. Initial side: negative \( y \)-axis, terminal side: positive \( x \)-axis, counterclockwise rotation.
3. Initial side: positive \( y \)-axis, terminal side: positive \( y \)-axis, counterclockwise rotation.
4. Initial side: negative \( x \)-axis, terminal side: negative \( y \)-axis, clockwise rotation.

For the given angles, state (a) whether it is acute, obtuse or neither, (b) its supplement, (c) its complement, if it is defined, and (d) its quadrant, if it is defined.

5. \( 43^\circ \)
6. \( 71^\circ \)
7. \( 120^\circ \)
8. \( 235^\circ \)
9. \( 90^\circ \)
10. \( 180^\circ \)
11. \( 35.26^\circ \)
12. \( 174.18^\circ \)

For the given angles, state its equivalent coterminal angle as described

13. \( 30^\circ \), negative rotation.
14. \( 41^\circ \), negative rotation.
15. \( 52^\circ \), one extra positive rotation.
16. \( 112^\circ \), one extra positive rotation.
17. \( 230^\circ \), one extra negative rotation.
18. \( 310^\circ \), one extra negative rotation.
19. \( 67.2^\circ \), two extra positive rotations.
20. \( 31.4^\circ \), three extra negative rotations.

21. A basketball makes 5 complete rotations between the time it is let go by the shooter and before it enters the basket. How many degrees did the basketball turn while in the air?

22. A figure skater makes 8 complete rotations plus an extra half-rotation during part of her routine. How many degrees did she rotate?
23. Two angles are complementary, and one angle is 4 times as large as the smaller angle. State the two angles.

24. Two angles are complementary, and the smaller angle is 1/5 as big as the larger angle. State the two angles.

25. Two angles are supplementary, and the larger angle is 3.4 times as big as the smaller angle. State the two angles.

26. Two angles are supplementary, and the larger angle is 5 degrees less than twice the size of the smaller angle. State the two angles.

Convert the following angles from degrees into radian measure.

27. 30°
28. 40°
29. 135°
30. 225°
31. 300°
32. 330°
33. 10°
34. 36°
35. 144°
36. 200°
37. 450°
38. 720°

Convert the following angles from radian measure into degrees.

39. \(\frac{\pi}{9}\)
40. \(\frac{\pi}{12}\)
41. \(\frac{5\pi}{3}\)
42. \(\frac{\pi}{4}\)
43. \(\frac{13\pi}{24}\)
44. \(\frac{19\pi}{36}\)
45. \(\frac{9\pi}{5}\)
46. \(\frac{41\pi}{72}\)
47. \(3\pi\)
48. \(\frac{5\pi}{3}\)
49. \(\frac{11\pi}{3}\)
50. \(\frac{13\pi}{4}\)

51. State the complement and supplement to \(\frac{\pi}{4}\) in radian measure.
52. State the complement and supplement to $\frac{2\pi}{5}$ in radian measure.

53. How many degrees is 1 radian?

54. An angle has a radian measure of 4 radians. Which quadrant is this angle located in?

55. An angle sweeps out $\frac{1}{10}$th of a circle in the positive direction. What is this angle’s radian measure?

56. An angle sweeps out $\frac{2}{5}$th of a circle in the positive direction. What is this angle’s radian measure?

In the following problems, a ray is drawn at the given angle, intersecting the unit circle at point $P$. State $P$'s coordinates and the ray’s slope. Use a calculator and state your answers to three decimal places.

57. $38^\circ$
58. $71^\circ$
59. $124^\circ$
60. $176^\circ$
61. $201^\circ$
62. $266^\circ$
63. $292^\circ$
64. $344^\circ$

Use the Pythagorean Identity, quadrants and symmetry to answer the following questions. State your answers to three decimal places.

65. If $\sin \theta = 0.215$ and $\theta$ is in Quadrant 1, find $\cos \theta$ and $\tan \theta$.
66. If $\sin \theta = 0.702$ and $\theta$ is in Quadrant 1, find $\cos \theta$ and $\tan \theta$.
67. If $\cos \theta = -0.386$ and $\theta$ is in Quadrant 2, find $\sin \theta$ and $\tan \theta$.
68. If $\cos \theta = -0.445$ and $\theta$ is in Quadrant 2, find $\sin \theta$ and $\tan \theta$.
69. If $\sin \theta = -0.095$ and $\theta$ is in Quadrant 3, find $\cos \theta$ and $\tan \theta$.
70. If $\cos \theta = -0.166$ and $\theta$ is in Quadrant 3, find $\sin \theta$ and $\tan \theta$.
71. If $\cos \theta = 0.521$ and $\theta$ is in Quadrant 4, find $\sin \theta$ and $\tan \theta$.
72. If $\sin \theta = -0.861$ and $\theta$ is in Quadrant 4, find $\cos \theta$ and $\tan \theta$.

Use reference angles and symmetry to answer the following questions in exact form.

73. $\sin \frac{2\pi}{3}$
74. $\sin \frac{7\pi}{6}$
75. $\cos \frac{4\pi}{3}$
76. $\cos \frac{17\pi}{6}$
77. $\sin \frac{25\pi}{3}$
78. \( \cos \frac{31\pi}{6} \)  
79. \( \cos \frac{40\pi}{3} \)  
80. \( \sin \frac{99\pi}{4} \)  

Determine the remaining five trigonometric function values in exact form, based on the given information.

81. \( \sin \theta = \frac{1}{4}, \theta \) is in Quadrant 1.  
82. \( \sin \theta = \frac{2}{11}, \theta \) is in Quadrant 1.  
83. \( \cos \theta = -\frac{2}{3}, \theta \) is in Quadrant 2.  
84. \( \sin \theta = \frac{7}{11}, \theta \) is in Quadrant 2.  
85. \( \tan \theta = \frac{3}{8}, \theta \) is in Quadrant 3.  
86. \( \sin \theta = -\frac{3}{13}, \theta \) is in Quadrant 3.  
87. \( \sin \theta = -\frac{2}{9}, \theta \) is in Quadrant 4.  
88. \( \tan \theta = -5, \theta \) is in Quadrant 4.

Use your calculator to determine these trigonometric values to three decimal places. Note: most calculators do not have specific keys for the cotangent, secant and cosecant.

89. \( \cot 52^\circ \)  
90. \( \cot 134^\circ \)  
91. \( \csc 211^\circ \)  
92. \( \csc 318^\circ \)  
93. \( \sec 140^\circ \)  
94. \( \sec 75^\circ \)  

95. If \( \sin \theta = \frac{1}{8} \) and \( \theta \) is in Quadrant 1, determine \( \sin 2\theta \) and \( \cos 2\theta \).

96. If \( \cos \theta = -\frac{1}{3} \) and \( \theta \) is in Quadrant 2, determine \( \sin 2\theta \) and \( \cos 2\theta \).

97. Use a sum-difference formula to prove \( \sin 2\theta = 2 \sin \theta \cos \theta \).

98. Use a sum-difference formula to prove that \( \cos(2\theta) = \cos^2 \theta - \sin^2 \theta \)

99. The **Shift Identities** are

\[
\cos \left( \theta - \frac{\pi}{2} \right) = \sin \theta \\
\sin \left( \theta + \frac{\pi}{2} \right) = \cos \theta
\]

Use the sum-difference formulas to prove these two identities.
100. Use the sum-difference identities to show that $\sin(30) = \frac{1}{2}$, $\cos(30) = \frac{\sqrt{3}}{2}$ and $\tan(30) = \frac{\sqrt{3}}{3}$. Hint: $30 = 90 - 60$.

101. What is the period of the graph of $y = \sin 2\theta$?

102. What is the period of the graph of $y = \sin \frac{1}{3} \theta$?

103. The population of a mining town is modeled by the function

$$P(t) = 1500 - 300 \cos \left(\frac{\pi}{6} (t - 1)\right),$$

Where $t$ is months (January = 1, so forth). Your calculator should be in Radian mode.

a) Calculate the town’s population in March, July and November.
b) In what month is the population the lowest? The highest?