Basic Trigonometry You Should Know
(Not only for this class but also for calculus)

• Angle measurement: degrees and radians.

There are 360 degrees in a full circle. If the circle has radius 1, then the circumference is $2\pi$, and we call that the “radian” equivalent to 360 degrees. From this, we can take proportions and come up with many other common radian angle measurements. The most common are:

$30^\circ = \frac{\pi}{6} \text{ rad}, \ 45^\circ = \frac{\pi}{4} \text{ rad}, \ 60^\circ = \frac{\pi}{3} \text{ rad}, \ 90^\circ = \frac{\pi}{2} \text{ rad}, \ 180^\circ = \pi \text{ rad}, \ 270^\circ = \frac{3\pi}{2} \text{ rad}.$

• Converting between degrees and radians.

To convert from a degree measure to radian, multiply by $\frac{\pi}{180}$ and reduce the integer parts, leaving the $\pi$ alone. For example, the angle $36^\circ$ is converted into radians as such: $36\left(\frac{\pi}{360}\right) = \frac{\pi}{10}$ radians. To convert from radians into degrees, multiply by $\frac{180}{\pi}$. The $\pi$ will cancel, and reduce what remains. For example, $\frac{\pi}{9}$ radians is converted into degrees as follows: $\frac{\pi}{9}\left(\frac{180}{\pi}\right) = \frac{180}{9} = 20$ degrees.

• Arc length and radian measure.

When an angle $\theta$ radians “subtends” (cuts off) a portion of a circle of radius $r$, the length of the arc it creates is given by the formula $S = r\theta$, where $S$ is the arc length. Note: to use this formula, the angle must be in radians.

• The basic trigonometric functions and their meanings.

A unit circle (centered at the origin) is drawn, and a ray drawn from the origin passing through the circle at point $P$. If $\theta$ represents the angle of this ray, then the three basic trigonometric functions are defined as follows:

$\cos \theta = \text{x-coordinate of the point } P.$
$\sin \theta = \text{y-coordinate of the point } P.$
$\tan \theta = \text{slope of the ray}.$

• The Quadrants and the signs of the trigonometric functions.

The xy-plane is divided into four quadrants and the signs of the three principal trigonometric functions are as shown:

<table>
<thead>
<tr>
<th>Quadrant 2</th>
<th>Quadrant 1</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\cos \theta &lt; 0$</td>
<td>$\cos \theta &gt; 0$</td>
</tr>
<tr>
<td>$\sin \theta &gt; 0$</td>
<td>$\sin \theta &gt; 0$</td>
</tr>
<tr>
<td>$\tan \theta &lt; 0$</td>
<td>$\tan \theta &gt; 0$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Quadrant 3</th>
<th>Quadrant 4</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\cos \theta &lt; 0$</td>
<td>$\cos \theta &gt; 0$</td>
</tr>
<tr>
<td>$\sin \theta &lt; 0$</td>
<td>$\sin \theta &lt; 0$</td>
</tr>
<tr>
<td>$\tan \theta &gt; 0$</td>
<td>$\tan \theta &lt; 0$</td>
</tr>
</tbody>
</table>
• Two basic Identities.

The two most common identities are:

1) The Pythagorean Identity: \( \cos^2 \theta + \sin^2 \theta = 1 \).
2) The tangent identity: \( \tan \theta = \frac{\sin \theta}{\cos \theta} \).

• The periods of the trigonometric functions.

The sine and cosine functions both have period \( 2\pi \), meaning the output values start to repeat themselves every \( 2\pi \) units. The tangent function has a period of \( \pi \).

• The exact values of cosine, sine and tangent in the first quadrant.

The table is:

<table>
<thead>
<tr>
<th>Angle ( \theta ) (Degrees)</th>
<th>Angle ( \theta ) (Radians)</th>
<th>( \cos \theta )</th>
<th>( \sin \theta )</th>
<th>( \tan \theta )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>30°</td>
<td>( \pi/6 )</td>
<td>( \sqrt{3}/2 )</td>
<td>( 1/2 )</td>
<td>( \sqrt{3}/3 )</td>
</tr>
<tr>
<td>45°</td>
<td>( \pi/4 )</td>
<td>( \sqrt{2}/2 )</td>
<td>( \sqrt{2}/2 )</td>
<td>1</td>
</tr>
<tr>
<td>60°</td>
<td>( \pi/3 )</td>
<td>( 1/2 )</td>
<td>( \sqrt{3}/2 )</td>
<td>( \sqrt{3} )</td>
</tr>
<tr>
<td>90°</td>
<td>( \pi/2 )</td>
<td>0</td>
<td>1</td>
<td>undefined</td>
</tr>
</tbody>
</table>

• The graphs of the sine, cosine and tangent functions.

The three basic graphs are:

• Sketching a general “Sinusoidal” graph.

The general form is \( f(x) = A \sin(Bx) + D \), where \( A \) is amplitude, \( D \) is the midline and \( B \) affects the period, where the period is found by \( P = \frac{2\pi}{B} \). You can replace “\( \sin \)” with “\( \cos \)” as needed. Also, be able to work backwards, that is, develop a model based on a graph. A useful mnemonic is “MAP”: Midline, Amplitude and Period. Determine these values in this order and it makes the process go
easier. Some models also include a “phase shift” (a modified horizontal shift): \( f(x) = A \sin(Bx + C) + D \), where \( C \) is the phase shift (strictly speaking, the quotient \( C/B \) is the horizontal shift).

- **The cosine, sine and tangent functions defined from right triangles.**

  The three common trigonometric functions can also be defined in terms of the legs and hypotenuse of a right triangle:

  \[
  \sin \theta = \frac{\text{opposite}}{\text{hypotenuse}}, \quad \cos \theta = \frac{\text{adjacent}}{\text{hypotenuse}}, \quad \tan \theta = \frac{\text{opposite}}{\text{adjacent}}.
  \]

  Remember, “SOHCAHTOA”.

- **The reciprocal trigonometric functions**

  The reciprocal trigonometric functions are the secant, cosecant and cotangent:

  \[
  \sec \theta = \frac{1}{\cos \theta} = \frac{\text{hypotenuse}}{\text{adjacent}}, \quad \csc \theta = \frac{1}{\sin \theta} = \frac{\text{hypotenuse}}{\text{opposite}}, \quad \cot \theta = \frac{1}{\tan \theta} = \frac{\cos \theta}{\sin \theta} = \frac{\text{adjacent}}{\text{opposite}}.
  \]

  In calculus, the secant function is used often. The cosecant and cotangent functions are used less frequently. It’s best to rewrite them into their equivalent sine or cosine forms.

- **The sum and difference identities for sine and cosine.**

  They are:

  \[
  \sin(\alpha \pm \beta) = \sin \alpha \cos \beta \pm \cos \alpha \sin \beta \\
  \cos(\alpha \pm \beta) = \cos \alpha \cos \beta \mp \sin \alpha \sin \beta
  \]

  Note the sign “reversal” in the cosine case. These are usually proven by laborious algebra, using the distance formula.

- **The double-angle identities for sine and cosine.**

  These can be proven using the sum-difference identities above. They are:

  \[
  \sin(2\theta) = 2 \sin \theta \cos \theta \\
  \cos(2\theta) = \cos^2 \theta - \sin^2 \theta
  \]

- **The half-angle identities for sine and cosine.**

  These are found by solving the double-angle identities and using some algebra. They are:

  \[
  \sin \theta = \sqrt{\frac{1-\cos 2\theta}{2}}, \quad \text{and} \quad \cos \theta = \sqrt{\frac{1+\cos 2\theta}{2}}.
  \]

  Note that both forms contain cosine. Just the internal sign differs.
• The Pythagorean identities and their corollaries.

The Pythagorean Identity has many equivalent forms.

\[
\begin{align*}
\cos^2 \theta + \sin^2 \theta &= 1 \\
\cos^2 \theta &= 1 - \sin^2 \theta \quad \text{(subtraction of } \sin^2 \theta) \\
\sin^2 \theta &= 1 - \cos^2 \theta \quad \text{(subtraction of } \cos^2 \theta) \\
1 + \tan^2 \theta &= \sec^2 \theta \quad \text{(divide through by } \cos^2 \theta) \\
\tan^2 \theta &= \sec^2 \theta - 1 \quad \text{(subtraction of 1)} \\
\cot^2 \theta + 1 &= \csc^2 \theta \quad \text{(divide through by } \sin^2 \theta) \\
\cot^2 \theta &= \csc^2 \theta - 1 \quad \text{(subtraction of 1)}
\end{align*}
\]

• The shift identities.

They are:

\[
\begin{align*}
\cos \left( \theta - \frac{\pi}{2} \right) &= \sin \theta \\
\sin \left( \theta + \frac{\pi}{2} \right) &= \cos \theta
\end{align*}
\]

This is because the sine graph is the same as the cosine graph shifted \( \frac{\pi}{2} \) units to the right, while the cosine graph is the same as the sine graph shifted \( \frac{\pi}{2} \) units to the left.

• The inverse trigonometric functions.

These functions “undo” the three principal trigonometric functions. They are:

- The inverse sine (or arcsin) written \( f(x) = \sin^{-1} x \), where $-1 \leq x \leq 1$.
- The inverse cosine (or arccos) written \( f(x) = \cos^{-1} x \), where $-1 \leq x \leq 1$.
- The inverse tangent (or arctan) written \( f(x) = \tan^{-1} x \), where $-\infty < x < \infty$.

In calculus, the inverse sine and inverse tangent functions are used often. In fact, the inverse tangent function is quite popular, especially in Integration.

You should be able to “reverse” the table of exact values using these functions. For example, \( \tan^{-1} 1 = \frac{\pi}{4} \), since \( \tan \frac{\pi}{4} = 1 \). Another example: \( \sin^{-1} \frac{\sqrt{3}}{2} = \frac{\pi}{3} \), since \( \sin \frac{\pi}{3} = \frac{\sqrt{3}}{2} \).

Note: your calculator only gives back the “primary” answer: Quadrants 1 and 4 for the inverse sine, quadrants 1 and 2 for the inverse cosine and quadrants 1 and 4 for the inverse tangent.
The graphs of the inverse trigonometric functions.

Note: the inverse sine (arcsin) and inverse cosine (arccos) functions are finite in length: what you see above are their entire graphs. They “end” at their respective endpoints. The inverse tangent (arctan) function extends across all real numbers and has two horizontal asymptotes, \( \pm \frac{\pi}{2} \).

The inverse sine function has a range of \(-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}\), while the inverse cosine function has a range of \(0 \leq y \leq \pi\). The range of the inverse tangent function is \(-\frac{\pi}{2} < y < \frac{\pi}{2}\). Note that it does not include the endpoints.