The standard form for an exponential function is \( f(x) = a \cdot b^x \), where \( a \) is the “initial amount” corresponding to \( x = 0 \), and \( b \) is the base, also known as the “growth factor”. Furthermore, the base \( b \) can be written as \( b = 1 + r \), where \( r \) represents the percent change.

**Example:** Given the exponential function \( f(x) = 125(1.035)^x \), we can identify the initial amount as \( a = 125 \), the growth factor \( b = 1.035 \), and the percent change (growth rate) as 3.5%.

**Example:** In the function \( g(x) = 61(1.77)^x \), the initial amount is 61, the growth factor is 1.77 and the growth rate is 77%.

**Caution!!** Be aware of the subtle difference between growth factor and the growth rate.

♦ **Set I:** Given the following functions, identify the initial amount, the growth factor and the growth (percentage) rate. Answers are given at the end.

1. \( f(x) = 100(1.06)^x \)
2. \( g(x) = 56(1.0525)^x \)
3. \( h(x) = 152(1.15)^x \)
4. \( k(t) = 48.2(2.5)^t \)
5. \( m(t) = 73(3)^t \)
6. \( P(t) = 1,200(0.97)^t \)
7. \( A(t) = 500(0.08)^t \)
8. \( M(t) = 720\left(\frac{2}{3}\right)^t \)

A table of values is sometimes given, from which we may determine the model.

**Example:**

<table>
<thead>
<tr>
<th>( x )</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>( y )</td>
<td>5</td>
<td>15</td>
<td>45</td>
<td>135</td>
</tr>
</tbody>
</table>

In this case, the initial amount is given—it corresponds to when \( x = 0 \); therefore, \( a = 5 \). We also see that each \( y \)-value is found by multiplying by 3 each time. Thus, the base (growth factor) is \( b = 3 \). The model is therefore \( y = 5(3)^x \). The growth rate is 200%.

**Example:**

<table>
<thead>
<tr>
<th>( x )</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>( y )</td>
<td>20</td>
<td>10</td>
<td>5</td>
<td>2.5</td>
</tr>
</tbody>
</table>

The initial amount is \( a = 20 \). The base is found by taking two consecutive outputs and finding their ratio, 2\(^{nd}\) number divided by 1\(^{st}\) number always: we see that we multiply by \( b = \frac{1}{2} \) each time. Therefore the model is \( y = 20\left(\frac{1}{2}\right)^x \). The growth rate is –50%.
Example:

<table>
<thead>
<tr>
<th></th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>x</td>
<td>20</td>
<td>80</td>
<td>320</td>
<td></td>
</tr>
<tr>
<td>y</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Here, we don’t know the initial amount. But we can see we are multiplying by 4 each time, so the base is 4. We would then divide to move left, so to speak. Therefore, it makes sense that the initial amount is $a = 5$, and that the whole model is $y = 5(4)^x$. The growth rate is 300%.

Example:

<table>
<thead>
<tr>
<th></th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>x</td>
<td>2</td>
<td>10</td>
<td>50</td>
<td>250</td>
<td>1250</td>
</tr>
<tr>
<td>y</td>
<td>6</td>
<td>12</td>
<td>24</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

There aren’t two consecutive outputs to determine $b$ directly, but we see that 10 multiplied by $b$ three times would give 1,250. Therefore, we can write $10b^3 = 1250$, and solve for $b$ using normal algebra. Dividing out by 10, we get $b^3 = 125$. Raising both sides to the $1/3$rd power, we get $b = 5$. The model is $y = 2(5)^x$. The growth rate is 400%, and the complete table is

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<td>4</td>
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<tr>
<td>y</td>
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<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>x</td>
<td>0</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
</tr>
<tr>
<td>y</td>
<td>...</td>
<td>16</td>
<td>32</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

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<td>0</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
</tr>
<tr>
<td>y</td>
<td>6</td>
<td>12</td>
<td>24</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Set II: Given the following tables, determine the model and fill in the remaining values.

1. $\begin{array}{cccc}
   x & 0 & 1 & 2 & 3 \\
   y & 6 & 12 & 24 & ...
\end{array}$
2. $\begin{array}{cccc}
   x & 0 & 1 & 2 & 3 \\
   y & ... & 16 & 32 & ...
\end{array}$
3. $\begin{array}{cccc}
   x & 0 & 1 & 2 & 3 \\
   y & ... & ... & 1320 & 330
\end{array}$
4. $\begin{array}{cccc}
   x & 0 & 1 & 2 & 3 & 4 \\
   y & ... & 5.32 & ... & 291.91904
\end{array}$

Applications

Example: A colony of bacteria doubles every 6 hours. If there were 100 bacteria to start with, set up a model that relates the population $P$ as a function of hours $t$.

Answer: We know that in 6 hours, there will be 200 bacteria in the colony. Therefore, 100 times the base to the $6^{th}$ power would give us 200. In other words, $100(b)^6 = 200$. Dividing by 100 we get $b^6 = 2$, then taking the $6^{th}$ root (1/$6^{th}$ power), we get $b = 2^{1/6} = 1.122$. The model is $P(t) = 100(1.122)^t$. The growth factor is 1.122, and the growth rate is 12.2% per hour.
Example: The population of a town doubles every 12 years. In 1990 there were 10,000 people living in the town. Set up a model and forecast the population of this town in 2011.

Answer: We have 1,000 people growing to 20,000 people in 12 years. This gives us the equation $b^{12} = 2$. Therefore, $b = 2^{1/12} = 1.059$. The model is $P(t) = 10,000(1.059)^t$, where $t = 0$ corresponds to the year 1990. Therefore, the year 2011 means that $t = 21$. The population in 2011 would be $10,000(1.059)^{21}$, which is about 33,328 people.

Example: A town had 4,000 people in 1992 and 7,000 people in 2003. Assuming exponential growth, what was the town’s population in (a) 2009? (b) 1987?

Answer: The starting point can be wherever you decide as long as you stay consistent with it. Thus, we can let 1992 be our “starting point”, so we declare that $t = 0$ represents 1992. This means that $t = 11$ represents 2003. We went from 4,000 people to 7,000 people in 11 years. This gives the equation $4,000b^{11} = 7,000$. Divide out by 4,000 and we have $b^{11} = \frac{7}{4}$, then take the $11^{th}$ root ($1/11^{th}$ power) to get $b: b = \left(\frac{7}{4}\right)^{1/11} = 1.052$. The model is $P(t) = 4,000(1.052)^t$. In 2009, there were $P(17) = 4,000(1.052)^{17} = 9,470$ people. In 1987, there were $P(-5) = 4,000(1.052)^{-5} = 3,104$ people.

Note: If you had started everything at 2003, you’re model would be slightly different but your answers to parts (a) and (b) would be the same, allowing for reasonable rounding error.

Example: An element loses 10% of its mass every 3 hours. How much remains after 1 day?

Answer: It appears not enough information is provided. However, there is. If we let $A$ represent the original quantity of the element (when $t = 0$), then in three hours, when $t = 3$, we know that 90% remains, which is written $0.9A$. Therefore, we can write $A \cdot b^3 = 0.9A$. The $A$'s divide out, leaving us with $b^3 = 0.9$. Take the third root ($1/3^{rd}$ power) and we have $b = 0.9^{1/3} = 0.965$. The model is $P(t) = A(0.965)^t$. Since no initial amount was given, we leave the $A$ as is. This is okay. After one day, we have $t = 24$, so $P(24) = A(0.965)^{24} = 0.425A$, which means about 42.5% of the original mass remains.

♦ Set III: Answer the following questions.

1. Suppose $500 is invested. Exactly 7 years later there is now $800 in the account. Set up an exponential model and state the (a) growth rate and (b) how much will be in the account after 20 years.
2. A town is losing population. In 2000 it has 1,200 people. In 2005 it had 950 people. Set up a model and state the (a) growth/decay rate, (b) the population in 1996 and (c) the population in 2011.
3. The population of a colony of bacteria triples every 9 hours. Suppose there were 30 milligrams to begin with. Set up a model and state the (a) growth rate, (b) the number in the colony after 5 hours, and (c) the number in the colony after a full day.
4. A town doubles in population every 25 years. What is the approximate percent change over a ten year period?
Answers

Set I

1. \( a = 100; b = 1.06; r = 6\% \)
2. \( a = 56; b = 1.0525; r = 5.25\% \)
3. \( a = 142; b = 1.15; r = 15\% \)
4. \( a = 48.2; b = 2.5; r = 150\% \)
5. \( a = 73; b = 3; r = 200\% \)
6. \( a = 1,200; b = 0.97; r = -3\% \)
7. \( a = 500; b = 0.08; r = -92\% \)
8. \( a = 720; b = \frac{2}{3}; r = -33.33\% \)

Set II

1. \( x \quad 0 \quad 1 \quad 2 \quad 3 \quad 4 \quad y = 6(2)^x \)
   \[
   \begin{array}{cccc}
   x & 0 & 1 & 2 & 3 & 4 \\
   y & 6 & 12 & 24 & 48 & 96 \\
   \end{array}
   \]
2. \( x \quad 0 \quad 1 \quad 2 \quad 3 \quad 4 \quad y = 8(2)^x \)
   \[
   \begin{array}{cccc}
   x & 0 & 1 & 2 & 3 & 4 \\
   y & 8 & 16 & 32 & 64 & 128 \\
   \end{array}
   \]
3. \( y = 5280 \quad 2640 \quad 1320 \quad 660 \quad 330 \quad y = 5280 \left(\frac{1}{2}\right)^x \)
   \[
   \begin{array}{cccc}
   x & 0 & 1 & 2 & 3 & 4 \\
   y & 5280 & 2640 & 1320 & 660 & 330 \\
   \end{array}
   \]
4. \( x \quad 0 \quad 1 \quad 2 \quad 3 \quad 4 \quad y = 1.4(3.8)^x \)
   \[
   \begin{array}{cccc}
   x & 0 & 1 & 2 & 3 & 4 \\
   y & 1.4 & 5.32 & 20.216 & 76.8208 & 291.91904 \\
   \end{array}
   \]

Set III

1. \( A(t) = 500(1.069)^t; \ r = 6.9\% \ \text{per year}; \ A(20) = $1,899 \)
2. \( P(t) = 1,200(0.954)^t; \ r = -4.6\% \ \text{per year}; \ P(-4) = 1,449; \ P(11) = 715. \) This all assumes that 2000 is our “starting point”. The values will be the same for any other starting point, allowing for rounding error.
3. \( A(t) = 30(1.130)^t; \ 13\% \ \text{per hour}; \ A(5) = 55 \ \text{(rounded)}; \ A(24) = 564 \ \text{(rounded)}. \)
4. \( P(t) = A(1.028)^t; \ P(10) = 1.318A. \) There is about a 31.8\% percent increase in the town’s population over a 10-year period.

Report any possible errors to me at surgent@asu.edu

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