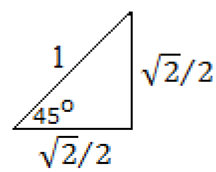
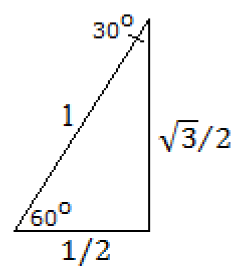
Exact Values of the Sine and Cosine Functions in Increments of 3 degrees

The sine and cosine values for all angle measurements in multiples of 3 degrees can be determined *exactly*, represented in terms of square-root radicals, and the four common operations of arithmetic.

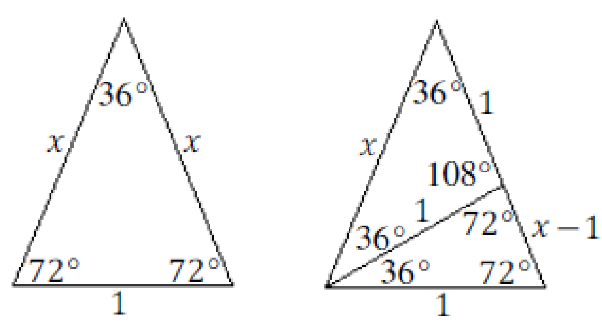
From the 45-45-90 degree triangle, set the hypotenuse to 1, and use the Pythagorean formula to determine the legs, which are both $\sqrt{2}/2$. Using the right-triangle constructions $\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}}$ and $\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}}$, we have $\sin 45° = \sqrt{2}/2$ and $\cos 45° = \sqrt{2}/2$.



From the equilateral (60-60-60) triangle with all legs of length 1, we can view half the triangle, forming a 30-60-90 triangle, with the hypotenuse measuring 1, the short leg $1/2$, and the long leg $\sqrt{3}/2$. Therefore, we have $\sin 30° = 1/2$, $\cos 30° = \sqrt{3}/2$, $\sin 60° = \sqrt{3}/2$ and $\cos 60° = 1/2$.



From the isosceles 36-72-72 triangle, let the two long sides be x, and the short side 1. Then, bisect one of the $72°$ angles, extending the ray to intersect the other side of the triangle. This forms two triangles: an isosceles 36-36-108 triangle whose short sides are both 1, and a smaller isosceles 36-72-72 triangle whose long sides are 1, and short side is $x - 1$, as shown in the diagrams below:



The two 36-72-72 triangles are proportional: The ratio of the long sides x: 1 is the same proportion as the ratio of the short sides: $1: (x - 1)$. Therefore, we solve for x by equating two ratios. We have $\frac{x}{1} = \frac{1}{x-1}$, which gives $x^2 - x = 1$ after cross-multiplying. Solving for x using the quadratic formula (and ignoring the negative root), we have $x = (1 + \sqrt{5})/2$. Splitting the 36-36-108 triangle in half, we now have a right triangle, a 36-54-90 triangle, with hypotenuse 1, long leg $x/2$ and short leg $\sqrt{1 - (x/2)^2}$. Since $x = (1 + \sqrt{5})/2$, the long leg of the 36-54-90 triangle is $(1 + \sqrt{5})/4$ and the short leg is $\sqrt{10 - 2\sqrt{5}}/4$.

Therefore, we can state the following:

$$\sin 36° = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{\sqrt{10-2\sqrt{5}}/4}{1} = \frac{\sqrt{10-2\sqrt{5}}}{4};\ \cos 36° = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{(1+\sqrt{5})/4}{1} = \frac{1+\sqrt{5}}{4}.$$

The remaining exact representations for angles of multiples of $3°$ can now be found, using sum-difference or half-angle identities. For example, $\sin 6° = 2 \sin 3° \cos 3°$, and so on. Since different identities may be used at various steps, the exact representations may "look" different than other possible representations, but can be shown to be identical in value. The following is a table of all exact values for the sine and cosine of angles of multiples of $3°$, up through $45°$. All radicals were simplified so that none contained any quotients within them.

Angle θ	$\sin\theta$	$\cos\theta$
0 degrees 0 radians	0	1
3 degrees $\frac{\pi}{60}$ radians	$\frac{1}{4}\sqrt{8-\sqrt{3}-\sqrt{15}-\sqrt{10-2\sqrt{5}}}$	$\frac{1}{4}\sqrt{8+\sqrt{3}+\sqrt{15}+\sqrt{10-2\sqrt{5}}}$
6 degrees $\frac{\pi}{30}$ radians	$\frac{1}{4}\sqrt{9-\sqrt{5}-\sqrt{30+6\sqrt{5}}}$	$\frac{1}{4}\sqrt{7+\sqrt{5}+\sqrt{30+6\sqrt{5}}}$
9 degrees $\frac{\pi}{20}$ radians	$\frac{1}{4}\sqrt{8-2\sqrt{10+2\sqrt{5}}}$	$\frac{1}{4}\sqrt{8+2\sqrt{10+2\sqrt{5}}}$
12 degrees $\frac{\pi}{15}$ radians	$\frac{1}{4}\sqrt{8-2\sqrt{8+\sqrt{30+6\sqrt{5}}-\sqrt{6-2\sqrt{5}}}}$	$\frac{1}{4}\sqrt{8+2\sqrt{8+\sqrt{30+6\sqrt{5}}-\sqrt{6-2\sqrt{5}}}}$
15 degrees $\frac{\pi}{12}$ radians	$\frac{1}{2}\sqrt{2-\sqrt{3}}$ or $\frac{1}{4}(\sqrt{6}-\sqrt{2})$	$\frac{1}{2}\sqrt{2+\sqrt{3}}$ or $\frac{1}{4}(\sqrt{6}+\sqrt{2})$
18 degrees $\frac{\pi}{10}$ radians	$\frac{1}{4}\sqrt{6-2\sqrt{5}}$	$\frac{1}{4}\sqrt{10+2\sqrt{5}}$

Angle θ	$\sin\theta$	$\cos\theta$
21 degrees $\frac{7\pi}{60}$ radians	$\frac{1}{4}\sqrt{8-\sqrt{10+2\sqrt{5}}-\sqrt{18-6\sqrt{5}}}$	$\frac{1}{4}\sqrt{8+\sqrt{10+2\sqrt{5}}+\sqrt{18-6\sqrt{5}}}$
24 degrees $\frac{2\pi}{15}$ radians	$\frac{1}{4}\sqrt{8-\sqrt{30+6\sqrt{5}}+\sqrt{6-2\sqrt{5}}}$	$\frac{1}{4}\sqrt{8+\sqrt{30+6\sqrt{5}}-\sqrt{6-2\sqrt{5}}}$
27 degrees $\frac{3\pi}{20}$ radians	$\frac{1}{4}\sqrt{8-2\sqrt{10-2\sqrt{5}}}$	$\frac{1}{4}\sqrt{8+2\sqrt{10-2\sqrt{5}}}$
30 degrees $\frac{\pi}{6}$ radians	$\frac{1}{2}$	$\frac{1}{2}\sqrt{3}$
33 degrees $\frac{11\pi}{60}$ radians	$\frac{1}{4}\sqrt{8-\sqrt{3}-\sqrt{15}+\sqrt{10-2\sqrt{5}}}$	$\frac{1}{4}\sqrt{8+\sqrt{3}+\sqrt{15}-\sqrt{10-2\sqrt{5}}}$
36 degrees $\frac{\pi}{5}$ radians	$\frac{1}{4}\sqrt{10-2\sqrt{5}}$	$\frac{1}{4}\left(1+\sqrt{5}\right)$
39 degrees $\frac{13\pi}{60}$ radians	$\frac{1}{4}\sqrt{8-\sqrt{10+2\sqrt{5}}+\sqrt{18-6\sqrt{5}}}$	$\frac{1}{4}\sqrt{8+\sqrt{10+2\sqrt{5}}-\sqrt{18-6\sqrt{5}}}$
42 degrees $\frac{7\pi}{30}$ radians	$\frac{1}{4}\sqrt{9+\sqrt{5}-\sqrt{30-6\sqrt{5}}}$	$\frac{1}{4}\sqrt{7-\sqrt{5}+\sqrt{30-6\sqrt{5}}}$
45 degrees $\frac{\pi}{4}$ radians	$\frac{1}{2}\sqrt{2}$	$\frac{1}{2}\sqrt{2}$

For sine and cosine measurements above $45°$ $\left(\frac{\pi}{4}\text{ radians}\right)$, use the identities $\cos\theta = \sin(90° - \theta)$ and $\sin\theta = \cos(90° - \theta)$. For example, $\sin 48° = \cos 42°$, and so on.

All of these values are *algebraic numbers*, meaning they are the root of some polynomial with rational coefficients. For example, $\sin 90°$ is algebraic. It equals 1, which is the root of the polynomial $x - 1$. Given an algebraic number in the form of radicals and arithmetic operations such as those listed in the above table, one can build a polynomial for which the given number is a root. For example, if $x = \frac{1}{2}\sqrt{2}$, then squaring both sides and collecting terms to one side, we get $x^2 - \frac{1}{2} = 0$, and so $2x^2 - 1$ is a polynomial for which $\frac{1}{2}\sqrt{2}$ is a root.

Building such polynomials is easy, but tedious. For example, $\sin 9° = \frac{1}{4}\sqrt{8 - 2\sqrt{10 + 2\sqrt{5}}}$, and thus is a root of the polynomial $1024x^8 - 2048x^6 + 1216x^4 - 192x^2 + 4$. Interestingly, this polynomial also has $\cos 9°$ as a root, too. Is that surprising? Or not?

The Interesting Case of Sin 10° and the Polynomial $8x^3 - 6x + 1 = 0$

The expression $\sin(3\theta)$ can be written as $-4\sin^3\theta + 3\sin\theta$ by observing that $\sin(3\theta) = \sin(2\theta + \theta)$ and using the sum identities. If we let $\theta = 10°$, we have

$$\sin(30°) = -4\sin^3(10°) + 3\sin(10°), \text{ or } \frac{1}{2} = -4\sin^3(10°) + 3\sin(10°).$$

Multiplying by 2, we have

$$1 = -8\sin^3(10°) + 6\sin(10°).$$

Thus, the polynomial $8x^3 - 6x + 1 = 0$ has sin (10°) as a root, and thus sin (10°) is algebraic. Graphing this polynomial and using the "zero" feature, the three roots of this polynomial are (rounded) $x = 0.1736482, x = 0.7660444$ and $x = -0.9396926$. A calculator also shows that $\sin(10°) = 0.1736482$. What are these other values?

Since sin (10°) is a root of $8x^3 - 6x + 1 = 0$, we factor this polynomial by dividing by $(x - \sin(10°))$ using synthetic division. The first pass through gives

$$8x^3 - 6x + 1 = (x - \sin(10°))\left(8x^2 + (8\sin(10°))x + (8\sin^2(10°) - 6)\right).$$

The remaining quadratic can be solved using the quadratic formula:

$$x = \frac{-8\sin(10°) \pm \sqrt{(8\sin(10°))^2 - 4(8)(8\sin^2(10°) - 6)}}{2(8)}.$$

After considerable simplification, we get

$$x = -\frac{1}{2}\left(\sin(10°) \pm \sqrt{3}\cos(10°)\right).$$

A calculator verifies that $-\frac{1}{2}\left(\sin(10°) + \sqrt{3}\cos(10°)\right) \approx -0.9396926$ ..., and that $-\frac{1}{2}\left(\sin(10°) - \sqrt{3}\cos(10°)\right) \approx 0.76604444$ ....

We can also try factoring this cubic directly. This is a depressed cubic (missing its quadratic term). It is known that the closed form solution will consist of cube roots of complex numbers, even though the answer is clearly real. Nevertheless, it's interesting to explore this avenue. The methods here date back to

Tartaglia, del Ferro, Cardan and his contemporaries of the 16th- and 17th-Centuries. First, let $x = r - \frac{k}{r}$, where r is a temporary variable and k is some constant to be determined:

$$8\left(r - \frac{k}{r}\right)^3 - 6\left(r - \frac{k}{r}\right) + 1 = 0.$$

Expanding, we have

$$8r^3 - 24rk + \frac{24k^2}{r} - \frac{k^3}{r^3} - 6r + \frac{6k}{r} + 1 = 0.$$

Note what happens when $k = -1/4$:

$$\begin{aligned} 8r^3 - 24r(-1/4) + \frac{24(-1/4)^2}{r} - 8\frac{(-1/4)^3}{r^3} - 6r + \frac{6(-1/4)}{r} + 1 = 0 \\ 8r^3 + 6r + \frac{3}{2}r + \frac{1}{8r^3} - 6r - \frac{3}{2}r + 1 = 0 \\ 8r^3 + \frac{1}{8r^3} + 1 = 0. \end{aligned}$$

Multiplying through by $8r^3$, we have

$$64r^6 + 8r^3 + 1 = 0.$$

This is quadratic if we treat $r^6 = (r^3)^2$. Using the quadratic formula, we have

$$r^3 = \frac{-8 \pm \sqrt{8^2 - 4(1)(64)}}{2(64)} = \frac{-8 \pm 8i\sqrt{3}}{128} = \frac{-1 \pm i\sqrt{3}}{16}.$$

Thus,

$$r = \left(\frac{-1 \pm i\sqrt{3}}{16}\right)^{1/3}.$$

Since $x = r + \frac{1}{4r}$, we have

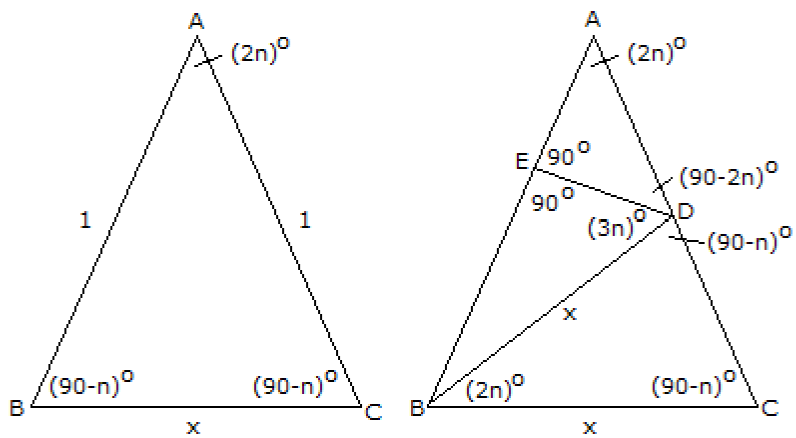
$$x = \left(\frac{-1 \pm i\sqrt{3}}{16}\right)^{1/3} + \frac{1}{4}\left(\frac{-1 \pm i\sqrt{3}}{16}\right)^{-1/3}.$$

A calculator shows that $\left(\frac{-1+i\sqrt{3}}{16}\right)^{1/3} + \frac{1}{4}\left(\frac{-1+i\sqrt{3}}{16}\right)^{-1/3} \approx 0.76604444$ .... Interestingly, the value of $\left(\frac{-1-i\sqrt{3}}{16}\right)^{1/3} + \frac{1}{4}\left(\frac{-1-i\sqrt{3}}{16}\right)^{-1/3}$ is also 0.76604444… Thus, it is clear (in a manner of speaking) that $\left(\frac{-1+i\sqrt{3}}{16}\right)^{1/3} + \frac{1}{4}\left(\frac{-1+i\sqrt{3}}{16}\right)^{-1/3} = -\frac{1}{2}\left(\sin(10°) - \sqrt{3}\cos(10°)\right)$.

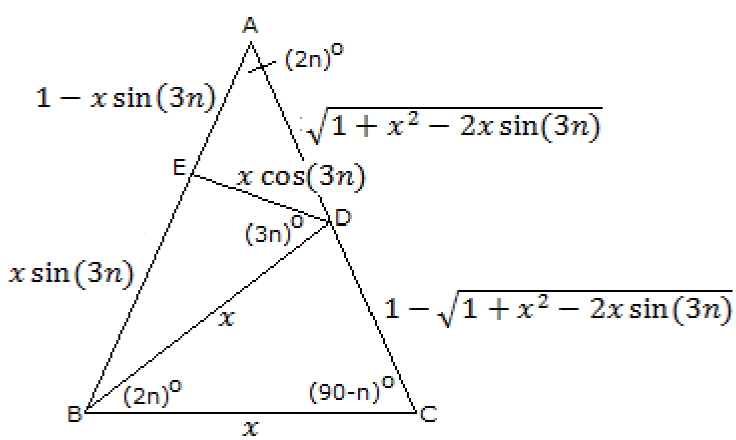
Determining the cube roots of complex numbers requires more wrestling with deMoivre's Theorem. Let's move on.

What About the Sine of 1 degree?

The following construction results in a general way to illustrate the value of $\sin n°$. Start with an isosceles triangle ΔABC with two sides of length 1, and the remaining side of length x. Let the angle opposite x be $(2n)°$.



In the above diagram, continue the construction as follows: Draw segment BD such that its length is also x, then draw segment DE such that it meets segment AB at a right angle. We can now label the various lengths as follows: $|DE| = x\cos(3n)$ and $|BE| = x\sin(3n)$, so therefore, $|AE| = 1 - x\sin(3n)$. The Pythagorean formula gives the length of $|AD| = \sqrt{1 + x^2 - 2x\sin(3n)}$. In turn, the length $|CD| = 1 - \sqrt{1 + x^2 - 2x\sin(3n)}$. This is shown in the following figure:



Now, drop a perpendicular from A to segment BC, and also a perpendicular from B to segment CD. In doing so, we have split the angle measurement $(2n)°$ into $n°$. Importantly, note that triangles ΔABC and ΔBCD are proportional. We can now define $\sin n°$ in two ways using the "opposite over hypotenuse" construction for right angles:

$$\sin n° = \frac{x}{2} \text{ and } \sin n° = \frac{1 - \sqrt{1 + x^2 - 2x\sin(3n)}}{2x}.$$

Relating the two expressions, we have:

$$\frac{1-\sqrt{1+x^2-2x\sin(3n)}}{2x}=\frac{x}{2}.$$

This simplifies to $x^2 = 1-\sqrt{1+x^2-2x\sin(3n)}$. After squaring away the radical, the equation becomes

$$x^4-3x^2+2x\sin(3n)=0$$

Since $x=0$ produces a trivial case, we ignore it and divide out by x:

$$x^3-3x+2\sin(3n)=0$$

This cubic polynomial has three roots. Let a be the positive root of this polynomial that is closest to 0. Therefore, $\sin(n^\circ)=\frac{a}{2}$, or $a=2\sin(n^\circ)$. For example, to find $\sin 1^\circ$, let $n=1$ and we get have $x^3-3x+2\sin(3^\circ)=0$. A calculator shows that $a=0.0349048\ldots$ is a root of this polynomial. Therefore, $\sin 1^\circ=\frac{0.0349048}{2}=0.0174524$, which is also confirmed via calculator.

Prepared by Scott Surgent (surgent@asu.edu) Please report errors to me if you see one. Updated 10-16-2013
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