Integral transforms on $\mathbb{R}_+$ with kernels given by Meijer’s G-functions

Wolfgang zu Castell

Motivated by the study of D.St. Richard’s generalized Bessel functions, we investigate the recursively defined kernels

$$F_{d}^{(\alpha)}(\rho; \mathbf{x}) = \frac{2}{\alpha} \rho^{\frac{1}{\alpha}} \int_0^1 (1-t)^{\frac{1}{\alpha}-1} \cos [(1-t)\rho x_d] F_{d-1}^{(\alpha)}(\rho t; \mathbf{x}') \, dt, \quad \mathbf{x} = (\mathbf{x}', x_d) \in \mathbb{R}^d,$$

with

$$F_{1}^{(\alpha)}(\rho; x) = \frac{2}{\alpha} \rho^{\frac{1}{\alpha}-1} \cos (\rho x), \quad x \in \mathbb{R}, \quad \rho > 0, \quad 0 < \alpha \leq 1.$$ 

These kernels can be seen as a modification of Richard’s Bessel functions which are subsumed by setting $\alpha = 1$. For this case H. Berens & Y. Xu derived a representation in terms of the $(d-1)$st divided difference at the knots $x_1^2, \ldots, x_d^2, \mathbf{x} = (x_1, \ldots, x_d) \in \mathbb{R}^d$ of a one-dimensional function. We generalize their approach by using Dirichlet averages defined by B.C. Carlson, to get a similar representation in terms of a generalized divided difference.

Looking at the integral transform

$$\Phi(\mathbf{x}) = \int_0^\infty \varphi(\rho) F_{d}^{(\alpha)}(\rho; \mathbf{x}) \, d\rho, \quad \mathbf{x} \in \mathbb{R}^d,$$

we will then write the kernel $F_{d}^{(\alpha)}(\cdot; \mathbf{x})$ as scale mixture of the kernel

$$h_{d}^{(\alpha)}(\xi) = k_{\alpha,d} \xi^{-\frac{d+2}{\alpha}} G_{1,3}^{1,1} \left( \frac{\xi^2}{4}, \frac{d}{\alpha} - 1, 1, \frac{d-1}{2} \right), \quad \xi \in \mathbb{R}^+,$$

$k_{\alpha,d}$ being some normalizing constant. This one-dimensional G-transform turns out to be at the core of the transform defined by the kernel $F_{d}^{(\alpha)}$.

If $\hat{\Phi}$ denotes the Fourier transform of $\Phi$, then

$$\hat{\Phi}(\mathbf{v}) = (|v_1| \cdots |v_d|)^{-\frac{d+1}{2}} \varphi(|\mathbf{v}|_1), \quad \mathbf{v} = (v_1, \ldots, v_d) \in \mathbb{R}^d;$$

i.e., the functions are related via the Riesz potentials as given above.

This extends results derived by Berens and the author for the case $\alpha = 1$. See [Result. Math. 34 (1998), 69-84] for the relevant references.