UNIFORM ASYMPTOTIC EXPANSIONS

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The method of steepest descent is probably the best known procedure for finding asymptotic behavior of integrals of the form

\[ I(\lambda) = \int_C g(z) e^{\lambda f(z)} dz, \]

where \( f(z) \) and \( g(z) \) are analytic functions, \( \lambda \) is a large positive parameter, and \( C \) is a contour in the \( z \)-plane. It was introduced by Debye (1909) in a paper concerning Bessel functions of large order. Debye’s basic idea is to deform the contour \( C \) into a new path of integration \( C' \) so that the following conditions hold:

(a) \( C' \) passes through one or more zeros of \( f'(z) \).

(b) the imaginary part of \( f(z) \) is constant on \( C' \).

If we write \( z = x + iy \) and

\[ f(z) = u(x, y) + iv(x, y) \]

and suppose that \( z_0 = x_0 + iy_0 \) is a zero of \( f'(z) \), then it is known that \( (x_0, y_0) \) is a saddle point of \( u(x, y) \) and the new curve \( v(x, y) = v(x_0, y_0) \) gives the steepest paths on the surface \( u = u(x, y) \) in the Cartesian space \((x, y, u)\). For simplicity, we shall assume that \( z_0 \) is a simple zero of \( f'(z) \) so that \( f''(z_0) \neq 0 \). On the steepest path \( C' \), we have

\[ f(z) = f(z_0) - t^2, \]

where \( t \) is real and usually increases monotonically to \(+\infty\). Changing variable from \( z \) to \( t \) gives

\[ I(\lambda) = e^{\lambda f(z_0)} \int_{-\infty}^{\infty} g(z) \frac{dz}{dt} e^{-\lambda t^2} dt. \]

By expanding \( f(z) \) into a Taylor series at \( z_0 \) and substituting it into (2), we have by reversion

\[ z - z_0 = \sqrt{\frac{2}{-f''(z_0)}} t + c_2 t^2 + \cdots \]

Thus, as a first approximation, we obtain

\[ I(\lambda) \sim g(z_0) e^{\lambda f(z_0)} \sqrt{\frac{2}{-f''(z_0)}} \int_{-\infty}^{\infty} e^{-\lambda t^2} dt, \]

which in turn yields

\[ I(\lambda) \sim g(z_0) e^{\lambda f(z_0)} \sqrt{\frac{-2\pi}{\lambda f''(z_0)}}. \]
If \( f(z) \) has more than one saddle point, then the full contribution to the asymptotic behavior of the integral \( I(\lambda) \) can be obtained by adding the contributions from all relevant saddle points. For instance, if \( f(z) \) has two simple saddle points, say \( z_+ \) and \( z_- \), then the asymptotic behavior of \( I(\lambda) \) is given by

\[
I(\lambda) \sim g(z_+) e^{\lambda f(z_+)} \sqrt{\frac{-2\pi}{\lambda f''(z_+)}} + g(z_-) e^{\lambda f(z_-)} \sqrt{\frac{-2\pi}{\lambda f''(z_-)}},
\]

We shall assume that \( \text{Re} f(z_+) = \text{Re} f(z_-) \), for otherwise one of the terms on the right-hand side of formula (4) will dominate the other. For a detailed discussion of the steepest descent method, we refer to Copson [2, Chapter 7] or Wong [6, Chapter II, Sec. 4].

The above situation is completely changed when the function \( f(z) \) is allowed to depend on an auxiliary parameter \( \alpha \); the very form of the asymptotic approximation in (4) changes when the two saddle points \( z_+ \) and \( z_- \) coalesce. To be more specific, we consider the integral

\[
I(\lambda, \alpha) = \int_C g(z) e^{\lambda f(z, \alpha)} dz,
\]

and suppose that there exists a critical value of \( \alpha \), say \( \alpha = \alpha_0 \), such that for \( \alpha \neq \alpha_0 \), the two distinct saddle points \( z_+ \) and \( z_- \) in (4) are of multiplicity 1, but at \( \alpha = \alpha_0 \), these two points coincide and give a single saddle point \( z_0 \) of multiplicity 2. Thus

\[
f_z(z_0, \alpha_0) = f_{zz}(z_0, \alpha_0) = 0, \quad f_{zzz}(z_0, \alpha_0) \neq 0,
\]

and

\[
f_z(z_+; \alpha) = f_z(z_-; \alpha) = 0, \quad f_{zz}(z_\pm; \alpha_0) \neq 0
\]

for \( \alpha \neq \alpha_0 \). Since \( z_\pm \to z_0 \) and hence \( f_{zz}(z_\pm; \alpha) \to 0 \) as \( \alpha \to \alpha_0 \), the approximation in (4) is not valid in a neighborhood of \( \alpha_0 \).

To obtain an asymptotic expansion for \( I(\lambda, \alpha) \) as \( \lambda \to \infty \), which holds uniformly for \( \alpha \) in a neighborhood of \( \alpha_0 \), Chester, Friedman and Ursell (1957) introduced in what is now regarded as a classic paper, the cubic transformation

\[
f(z, \alpha) = \frac{1}{3} u^3 - \zeta u + \eta,
\]

where \( \zeta \) and \( \eta \) are functions of \( \alpha \). These functions are determined by the condition that the transformation \( z \to u \) is one-to-one and analytic in a neighborhood of \( z_0 \) for all \( \alpha \) in a neighborhood of \( \alpha_0 \), i.e., in a neighborhood of the two saddle points. Making the transformation (6),

\[
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the integral in (5) is reduced to the canonical form

\begin{equation}
I(\lambda, \alpha) = e^{\lambda \eta} \int_{C^*} \varphi_0(u) e^{\lambda(u^{3/3} - \zeta u)} du,
\end{equation}

where \( C^* \) is the image of \( C \) and

\[ \varphi_0(u) = \frac{dz}{du}. \]

To obtain an asymptotic expansion for the last integral, we use a method of Bleistein (1967) and write

\begin{equation}
\varphi_0(u) = a_0 + b_0 u + (u^2 - \zeta) \psi_0(u),
\end{equation}

where the coefficients \( a_0 \) and \( b_0 \) can be determined by setting \( u = \pm \sqrt{\zeta} \) on two sides of the equation. Inserting (8) in (7) gives

\begin{equation}
I(\lambda, \alpha) = e^{\lambda \eta} \left[ V(\lambda^{2/3} \zeta) \frac{a_0}{\lambda^{1/3}} + V'(\lambda^{2/3} \zeta) \frac{b_0}{\lambda^{2/3}} + I_1(\lambda, \alpha) \right],
\end{equation}

where

\begin{equation}
V(\lambda) = \int_{C^*} e^{\lambda^{3/3} - \lambda^\nu} dv
\end{equation}

and

\begin{equation}
I_1(\lambda, \alpha) = \int_{C^*} (u^2 - \zeta) e^{\lambda(u^{3/3} - \zeta u)} \psi_0(u) du.
\end{equation}

For simplicity, let us assume that the coefficient \( \zeta \) in (6) is real, and that the contour \( C^* \) can be deformed into one which begins at \( \infty e^{-\pi i/3} \), passes through \( \sqrt{\zeta} \), and ends at \( \infty e^{\pi i/3} \). Thus, we have

\begin{equation}
V(\lambda) = 2\pi i \text{Ai}(\lambda),
\end{equation}

where \( \text{Ai}(\lambda) \) is the Airy function. To the integral \( I_1(\lambda, \alpha) \) we apply an integration by parts, and the result is

\begin{equation}
I_1(\lambda, \alpha) = \frac{1}{\lambda} \int_{C^*} \varphi_1(u) e^{\lambda(u^{3/3} - \zeta u)} du,
\end{equation}

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where $\varphi_1(u) = \psi'_0(u)$. In view of the factor $\frac{1}{4}$ in (13), it is anticipated that the integral $I_1(\lambda, \alpha)$ is of a lower asymptotic order than the first two terms on the right-hand side of equation (9). Hence, as a first approximation, we obtain from (9) and (12)

$$I(\lambda, \alpha) \sim 2\pi i e^{\lambda_0} \left[ \text{Ai} \left( \lambda^{2/3} \zeta \right) \frac{a_0}{\lambda^{1/3}} + \text{Ai}' \left( \lambda^{2/3} \zeta \right) \frac{b_0}{\lambda^{2/3}} \right].$$

The integral $I_1(\lambda, \alpha)$ is exactly of the same form as the one in (7). Hence, the above procedure can be repeated, and will lead to an infinite asymptotic expansion. A detailed discussion of this method can be found in Bleistein and Handelsman [1, Chapter 9] or Wong [6, Chapter VII].

In recent investigation of asymptotic behavior of some orthogonal polynomials, we have encountered situations in which there are two critical values of $\alpha$, say $\alpha_+$ and $\alpha_-$, such that for $\alpha \neq \alpha_\pm$ there are two distinct saddle points $z_+$ and $z_-$ of multiplicity 1, but at $\alpha = \alpha_\pm$, these two points coincide and give saddle points $\xi^{\pm}$ of multiplicity 2. Thus,

$$f_z(\xi^{\pm}, \alpha_\pm) = f_{zz}(\xi^{\pm}, \alpha_\pm) = 0, \quad f_{zzz}(\xi^{\pm}, \alpha_\pm) \neq 0,$$

and

$$f_z(z_+, \alpha) = f_z(z_-, \alpha) = 0, \quad f_{zz}(z_+, \alpha) \neq 0 \quad \text{for} \quad \alpha \neq \alpha_\pm.$$

The following examples provide concrete illustrations of such situations.

**EXAMPLE 1.** *Meixner polynomials* $m_n(x; \beta, c)$; see [3]. From their generating function, one can get the integral representation

$$\frac{1}{n!} m_n(x; \beta, c) = \frac{1}{2\pi i} \int_C e^{n f(z, \alpha)} \frac{dz}{z(1-z)^\beta},$$

where $x = n\alpha, \alpha \in (0, \infty),

$$f(z, \alpha) = \alpha \log \left( 1 - \frac{z}{c} \right) - \alpha \log(1 - z) - \log z$$

and $C$ is a circle centered at the origin with radius less than $\min (1, |c|)$.

**EXAMPLE 2.** *Meixner-Pollaczek polynomials* $M_n(x; \delta, \eta)$; see [4]. By the same argument, one also has the integral representation

$$\frac{1}{n!} M_n(x; \delta, \eta) = \frac{1}{2\pi i} \int_C \left[ (1 + \delta z)^2 + z^2 \right]^{-\eta/2} e^{n f(z, \alpha)} \frac{dz}{z},$$

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where $x = n\alpha, \alpha \in (0, \infty)$,

\[(18) \quad f(z, \alpha) = \alpha \tan^{-1}\left(\frac{z}{1 + \delta z}\right) - \log z \]

and $C$ is a circle centered at the origin with radius $1/\sqrt{1 + \delta^2}$. If we put

\[z_0 = -\delta + i \frac{1}{1 + \delta^2}\]

and

\[z_0 = r_0 e^{i\theta_0} \quad \text{with} \quad r_0 = \frac{1}{\sqrt{1 + \delta^2}},\]

then (18) can also be written as

\[(19) \quad f(z, \alpha) = \frac{\alpha}{2i} \log (z - z_0) - \frac{\alpha}{2i} \log(z - \bar{z}_0) - \log z + \alpha(\pi - \theta_0).\]

**Example 3.** Kravchouk polynomials $K_n^{(N)}(x;p,q)$; see [5]. They have the integral representation

\[K_n^{(N)}(x; p, q) = \frac{\eta^{p-n} \alpha^n}{2\pi i} \int_C e^{N f(z, \alpha)} \frac{dz}{z},\]

where $\sigma \equiv p/q, \alpha \equiv x/N, \nu \equiv n/N$,

\[(20) \quad f(z, \alpha) = (1 - \alpha) \log(1 - z) + \alpha \log(\sigma + z) - \nu \log z\]

and $C$ is a small closed contour surrounding $z = 0$.

In all three examples above, the large variable is $n$ (or $N$). A simple function which exhibits two saddle points coalescing at two distinct places is given by

\[(21) \quad \psi(u, \eta) = -\rho \log u + \eta \rho - \frac{u^2}{2}.\]

The saddle points occur at

\[(22) \quad u_{\pm} = \frac{\eta \pm \sqrt{\eta^2 - 4\rho}}{2},\]

and they coincide when $\eta = \pm 2\sqrt{\rho}$. If we put $d = -\lambda \rho - \frac{1}{2}(\rho > 0)$ and $z = \sqrt{\lambda} \eta$ in the integral representation of the parabolic cylinder function

\[(23) \quad U(d, z) = \frac{e^{-\frac{1}{2}d^2/4}}{2\pi i} \int_{-\infty}^{0^+} e^{z u - \frac{1}{2} u^2} u^{-\frac{1}{2}} du,\]
we obtain

\[ U(-\lambda \rho - \frac{1}{2}, \sqrt{\lambda} \eta) = \frac{(\lambda \rho + 1)}{2\pi i} e^{-\lambda \eta^2/4} \lambda^{-\lambda \rho/2} \int_{-\infty}^{(0+)} e^{\lambda \psi(u, \eta)} \frac{du}{u}. \]  

We now return to the integral \( I(\lambda, \alpha) \) in equation (5), and suppose that \( f(z, \alpha) \) satisfies the conditions in (15) and (16). To derive an asymptotic expansion for \( I(\lambda, \alpha) \), as \( \lambda \to \infty \), which holds uniformly in a region containing both \( \alpha_+ \) and \( \alpha_- \), we compare it with the integral in (24). This suggests that we make the transformation \( z \to u(z) \) defined by

\[ f(z, \alpha) = \psi(u, \eta) + \gamma, \]

where \( \gamma \) is a constant to be determined, and require \( u(0) = 0 \). Changing variable from \( z \) to \( u \), the integral in (5) becomes

\[ I(\lambda, \alpha) = e^{\lambda \gamma} \int_{-\infty}^{(0+)} \phi(u) e^{\lambda \psi(u, \eta)} \frac{du}{u}, \]

where

\[ \phi(u) = g(z) \frac{\psi_u(u, \eta)}{f_z(z, \alpha)} u. \]

The contour \( C \) in the \( z \)-plane should first be deformed into a steepest descent path; it will than be mapped into the loop path shown in (26) in the \( u \)-plane. Put \( \phi_0(u) = \phi(u) \), and write

\[ \phi_0(u) = a_0 + b_0 u + (u - u_+)(u - u_-)h_0(u), \]

where \( u_+ \) and \( u_- \) are given in (22). By setting \( u = u_+ \) and \( u = u_- \) on two sides of the equation, one finds that the coefficients \( a_0 \) and \( b_0 \) can be expressed in terms of \( \phi_0(u_+) \) and \( \phi_0(u_-) \). For simplicity, let us define the new function

\[ W(x, \lambda) \equiv e^{x^2/4} U(-\lambda \rho - \frac{1}{2}, x). \]

Clearly

\[ W(\sqrt{\lambda} \eta, \lambda) = \frac{(\lambda \rho + 1)}{2\pi i} e^{\lambda \eta^2/4} \lambda^{-\lambda \rho/2} \int_{-\infty}^{(0+)} e^{\lambda \psi(u, \eta)} \frac{du}{u}, \]

and from (23)

\[ W_x(\sqrt{\lambda} \eta, \lambda) = \frac{(\lambda \rho + 1)}{2\pi i} \lambda^{1/2-\lambda \rho/2} \int_{-\infty}^{(0+)} e^{\lambda \psi(u, \eta)} du. \]
Substituting (28) in (26) gives

\begin{equation}
I(\lambda, \alpha) = \frac{2\pi i}{(\lambda \rho + 1)} \lambda^{\rho/2} e^{\lambda \gamma} \left[ a_0 W(\sqrt{\lambda} \eta, \lambda) + \frac{b_0}{\sqrt{\lambda}} W_x(\sqrt{\lambda} \eta, \lambda) + \varepsilon_1 \right],
\end{equation}

where

\[ \varepsilon_1 = \frac{\lambda \rho + 1}{2\pi i} \lambda^{-\lambda \rho/2} \int_{-\infty}^{(0^+)} e^{\lambda \psi(u, n)} (u - u_+)(u - u_-) h_0(u) \frac{du}{u}. \]

An integration by parts gives

\[ \varepsilon_1 = \frac{1}{\lambda} \frac{(\lambda \rho + 1)}{2\pi i} \lambda^{-\lambda \rho/2} \int_{-\infty}^{(0^+)} e^{\lambda \psi(u, n)} \phi_1(u) \frac{du}{u}, \]

where \( \phi_1(u) = u h'_0(u) \). Neglecting the error term \( \varepsilon_1 \), we have from (30), as a first approximation,

\[ I(\lambda, \alpha) \sim \frac{2\pi i}{(\lambda \rho + 1)} \lambda^{\rho/2} e^{\lambda \gamma} \left[ a_0 W(\sqrt{\lambda} \eta, \lambda) + \frac{b_0}{\sqrt{\lambda}} W_x(\sqrt{\lambda} \eta, \lambda) \right]. \]

This process can again be repeated to yield an infinite asymptotic expansion.

REFERENCES


