Tridiagonal pairs

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ABSTRACT

We consider the following situation in linear algebra. Let $\mathcal{F}$ denote a field, and let $V$ denote a vector space over $\mathcal{F}$ with finite positive dimension. We consider a pair of linear transformations $A : V \to V$ and $A^* : V \to V$ satisfying the following four conditions.

1. $A$ and $A^*$ are both diagonalizable on $V$.

2. There exists an ordering $V_0, V_1, \ldots, V_d$ of the eigenspaces of $A$ such that

   $$A^* V_i \subseteq V_{i-1} + V_i + V_{i+1} \quad (0 \leq i \leq d),$$

   where $V_{-1} = 0, V_{d+1} = 0$.

3. There exists an ordering $V_0^*, V_1^*, \ldots, V_\delta^*$ of the eigenspaces of $A^*$ such that

   $$A V_i^* \subseteq V_{i-1}^* + V_i^* + V_{i+1}^* \quad (0 \leq i \leq \delta),$$

   where $V_{-1}^* = 0, V_{\delta+1}^* = 0$.

4. There is no subspace $W$ of $V$ such that both $A W \subseteq W, A^* W \subseteq W$, other than $W = 0$ and $W = V$.

We call such a pair a Tridiagonal pair, or TD pair, for short. We wish to classify all TD pairs. We are motivated by the following connection to orthogonal polynomials. By a Leonard pair, we mean a TD pair for which the eigenspaces $V_i$ and $V_i^*$ all have dimension 1. In [1], we showed there exists a 1–1 correspondence between the Leonard pairs and the finite length polynomial sequences of the Askey-Scheme. The most general polynomials of this sort are the $q$-Racah polynomials.

Concerning arbitrary TD pairs, we have the following results. Referring to the above TD pair, we show $d = \delta$. We show that for $0 \leq i \leq d$, the eigenspaces $V_i$ and $V_i^*$ have the same dimension. Denoting this common dimension by $\rho_i$, we show the sequence $\rho_0, \rho_1, \ldots, \rho_d$ is
symmetric and unimodal, i.e. \( \rho_{i-1} \leq \rho_i \) for \( 1 \leq i \leq d/2 \) and \( \rho_i = \rho_{d-i} \) for \( 0 \leq i \leq d \). We show that there exists a sequence of scalars \( \beta, \gamma, \gamma^*, \varrho, \varrho^* \) taken from \( \mathcal{F} \) such that both

\[
0 = [A, A^2 A^* - \beta A A^* A + A^* A^2 - \gamma (A A^* + A^* A) - \varrho A^*],
\]

\[
0 = [A^*, A^{*2} A - \beta A^* A A^* + A A^{*2} - \gamma^* (A^* A + A A^*) - \varrho^* A],
\]

where \([r, s] = rs - sr\). The sequence is unique if \( d \geq 3 \). Let \( \theta_i \) (resp. \( \theta_i^* \)) denote the eigenvalue of \( A \) (resp. \( A^* \)) associated with \( V_i \) (resp. \( V_i^* \)), for \( 0 \leq i \leq d \). We show the expressions

\[
\frac{\theta_{i-2} - \theta_{i+1}}{\theta_{i-1} - \theta_i}, \quad \frac{\theta_{i-2}^* - \theta_{i+1}^*}{\theta_{i-1}^* - \theta_i^*}
\]

both equal \( \beta + 1 \), for \( 2 \leq i \leq d - 1 \). We hope these results will ultimately lead to a complete classification of the TD pairs.

References

[1] P. Terwilliger. Two linear transformations each tridiagonal with respect to an eigenbasis of the other. Linear algebra and its applications, submitted