A ‘relativistic’ hypergeometric function obeying four Askey-Wilson type difference equations

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The hypergeometric function \( _2F_1(a, b; c; w) \) was already studied by Euler, using its integral representation involving powers of three functions linear in the integration variable. The subject of our lecture is a generalized hypergeometric function \( R(a_+, a_-, c_0, c_1, c_2, c_3; v, \hat{v}) \) that is defined in terms of an integral as well. Its integral representation does not generalize Euler’s \( _2F_1 \)-representation, however. Instead, our \( R \)-function representation reduces to the less well-known integral representation introduced by Barnes (about a century ago), when a suitable parameter limit is taken.

Barnes’ representation involves an integral whose building block is Euler’s gamma function. The gamma function asymptotics (Stirling’s formula) is crucial to establish that Barnes’ contour integral converges and that it defines \( _2F_1(a, b; c; w) \) as an analytic function of \( w \) in the cut plane \( \text{Arg}(w) \in (-\pi, \pi) \). Moreover, for \( |w| < 1 \) Stirling’s formula can be exploited to shift Barnes’ contour to infinity, picking up residues at simple poles of one of the gamma functions in the integrand. The resulting power series in \( w \) is then the well-known Gauss series for the hypergeometric function.

The integrand in the representation of the generalized hypergeometric function at issue is built from ‘hyperbolic gamma functions’, which generalize Euler’s ‘rational gamma function’. We have studied this hyperbolic gamma function \( G(a_+, a_-; z) \) several years ago in the context of a new approach to the theory of first order analytic difference equations, cf. Ref. [1]. We have occasion to exploit a number of its features in our account of the function \( R(a_+, a_-, c_0, c_1, c_2, c_3; v, \hat{v}) \). In particular, the asymptotics of the \( G \)-function plays a role analogous to Stirling’s for-
formula in our integral representation for the $R$-function. In contrast to the \( {}_2F_1 \)-function, however, we have no explicit power series representation available for our $R$-function.

In our lecture we concentrate on a sketch of various salient features of the $R$-function, viewed as a special function in its own right. Although we introduced it in order to diagonalize an analytic difference operator of relativistic Calogero-Moser type, it has far more general eigenfunction properties. Indeed, it is a joint eigenfunction of four independent analytic difference operators of Askey-Wilson type, two acting on $v$, and two on $\hat{v}$. When one of the variables $v, \hat{v}$ is suitably discretized, the $R$-function gives rise to polynomials that are basically the Askey-Wilson polynomials. This generalizes the discretization of $2F_1$ that gives rise to the Jacobi polynomials.

As we will sketch, in the ‘nonrelativistic limit’ $R \to {}_2F_1$ three of the four difference operators have a limit, too. One of the two difference operators acting on $v$ turns into the hyperbolic differential operator (‘nonrelativistic Schrödinger operator’) of which $2F_1$ is an eigenfunction (after the hyperbolic substitution $w \to -\sinh^2(v)$). The second one becomes a ‘free’ difference operator, whose action on $2F_1$ can be understood from the known continuation behavior across the logarithmic branch cut $w \in [1, \infty)$. The eigenfunction property of $2F_1$ with respect to the rational difference operator limit of the third hyperbolic difference operator (which acts on the dual variable $\hat{v}$) can be understood from the contiguous relations for $2F_1$.

Even though the hyperbolic gamma function is a crucial building block, we will not spend much time on its theory. Rather, we focus our lecture on the $R$-function, using as an input several properties of the $G$-function that are detailed in our paper Ref. [1]. In the same vein, we will be very short on the background of quantum integrable $N$-particle systems of relativistic Calogero-Moser type, from which the need for (and, accordingly, the introduction of) the $R$-function originated. We do indicate, however, in what sense the limit transition $R \to {}_2F_1$ can be viewed as a nonrelativistic limit.
We already introduced and discussed the $G$- and $R$-functions in our 1994 lecture notes Ref. [2], which deal with the integrable system context alluded to above. This survey and Ref. [1] constitute a rather self-contained background for our lecture. Another quite recent review of our work on generalized gamma functions, the ‘relativistic’ hypergeometric $R$-function, and ‘relativistic’ Lamé functions is Ref. [3]. Finally, our paper Ref. [4] may be consulted for a detailed account of the various aspects of the $R$-function sketched in our lecture.


