Stephen C. Milne
The Ohio State University, U.S.A.

Lecture 1
Transformations of $U(n+1)$ multiple basic hypergeometric series

Abstract

The purpose of this talk is to survey the transformation theory of $U(n+1)$ multiple basic hypergeometric series—starting with the $U(n+1)$ terminating very-well-poised $_6\phi_5$ summation theorems. These series were strongly motivated by L. C. Biedenharn and J. D. Louck and coworkers mathematical physics research involving angular momentum theory and the unitary groups $U(n+1)$, or equivalently $A_n$. They are directly related to the corresponding Macdonald identities. This $U(n+1)$ or $A_n$ theory has also been extended to the root systems $C_n$ and $D_n$. There are now many applications of $A_n$ and/or subsequently $C_n$ and $D_n$ multiple basic hypergeometric series. These include the following topics: A. N. Kirillov—quantum groups; R. Gustafson—multidimensional beta and/or Barnes integral evaluations; C. Krattenthaler and I. Gessel—plane partition enumeration; S. Milne—analytic number theory—sums of squares; S. Milne, G. Lilly, G. Bhatnagar, C. Krattenthaler, and M. Schlosser—multidimensional matrix inversions; S. Milne and V. Leininger—new infinite families of eta function identities; Y. Kajihara and M. Noumi—applications to raising operators for Macdonald polynomials (these operators were first introduced by A. N. Kirillov and M. Noumi, then by L. Lapointe and L. Vinet, and later by Y. Kajihara and M. Noumi.) As an introduction to this area we discuss some of the main results and techniques from the following outline of the development of $U(n+1)$ basic hypergeometric series.

The $U(n+1)$ terminating very-well-poised $_6\phi_5$ summation theorems extend Rogers’ classical one-variable work and are central to our theory. They may be proved using $q$-difference equations—arising from the Lagrange interpolation formula—and partial fraction expansions. The $U(n+1)$_6$\phi_5$ summation theorems may in turn be specialized to obtain the $U(n+1)$ extension of Andrews’ matrix formulation of the Bailey Transform. The $U(n+1)$ Bailey transform is then applied to the $U(n+1)$_6$\phi_5$ summation theorems to derive the $U(n+1)$ terminating, balanced $_3\phi_2$ summation theorems, whose special cases include $U(n+1)$_6$\phi_5$ summation theorems, $q$-Chu-Vandermonde theorems and $U(n+1)_q$-binomial theorems. An analytic continuation argument applied to a $U(n+1)_q$-binomial theorem yields the $U(n+1)$ extension of Ramanujan’s $_1\psi_1$ sum, and Gustafson’s $U(n+1)_6\psi_6$ summation turns out to be the next higher dimensional version of the $U(n+1)_1\psi_1$ sum. For example, the two-dimensional $U(3)_1\psi_1$ summation is equivalent to Bailey’s classical one-dimensional $_6\psi_6$ summation. We also obtain $U(n+1)$ extensions of the Jacobi triple product identity. The Bailey transform coupled with the $U(n+1)$ balanced $_3\phi_2$ summation theorems yields the $U(n+1)$ extension of Andrews’ explicit formulation of the
Bailey Lemma, which—upon iteration—gives several $U(n+1)$ generalizations of Watson’s $q$-analogue of Whipple’s transformation formula. Special and limiting cases include the non-terminating $U(n+1)\,_{6}\phi_{5}$ summation, the $U(n+1)$ extension of the terminating balanced $s\phi_{7}$ summation theorem, and the $U(n+1)$ Rogers-Ramanujan–Schur identities. A classical interchange of summation argument leads to the $U(n+1)\,_{10}\phi_{9}$ transformation formulas. Important limiting cases include the $U(n+1)$ generalization of Bailey’s non-terminating extension of Watson’s transformation. This in turn leads to the non-terminating $U(n+1)$ extension of Bailey’s balanced $s\phi_{7}$ summation theorem. The classical case of all this work, corresponding to $A_{1}$ or equivalently $U(2)$, contains a substantial amount of the theory and application of one-variable basic hypergeometric series.

**Lecture 2**

*Infinite families of exact sums of squares formulas, Jacobi elliptic Functions, continued fractions, and Schur functions*

**Abstract**

In this talk we give several infinite families of explicit exact formulas involving either squares or triangular numbers, two of which generalize Jacobi’s (1829) 4 and 8 squares identities to $4n^2$ or $4n(n+1)$ squares, respectively, without using cusp forms. (In fact, we similarly have generalized to infinite families all of Jacobi’s explicitly stated degree 2, 4, 6, 8 Lambert series expansions of classical theta functions. In addition, we have extended Jacobi’s special analysis of 2 squares, 2 triangles, 6 squares, 6 triangles to 12 squares, 12 triangles, 20 squares, 20 triangles, respectively.) These results, depending on new expansions for powers of various products of classical theta functions, arise in the setting of Jacobi elliptic functions, associated continued fractions, regular C-fractions, Hankel or Turánian determinants, Fourier series, Lambert series, inclusion/exclusion, Laplace expansion formula for determinants, and Schur functions. The Schur function form of these infinite families of identities are analogous to the $n$-function identities of Macdonald. Moreover, the powers $4n(n+1)$, $2n^2 + n$, $2n^2 - n$ that appear in Macdonald’s work also arise at appropriate places in our analysis. We also utilize a special case of our methods to give a proof of the two Kac-Wakimoto conjectured identities involving representing a positive integer by sums of $4n^2$ or $4n(n+1)$ triangular numbers, respectively. Our 16 and 24 squares identities were originally obtained via multiple basic hypergeometric series, Gustafson’s $C_t$ nonterminating $s\phi_{5}$ summation theorem, and Andrews’ basic hypergeometric series proof of Jacobi’s 2, 4, 6, and 8 squares identities. We have (elsewhere) applied symmetry and Schur function techniques to this original approach to prove the existence of similar infinite families of sums of squares identities for $n^2$ or $n(n+1)$ squares, respectively. Our sums of more than 8 squares identities are not the same as the formulas of Mathews (1895), Glaisher (1907), Ramanujan (1916), Mordell (1917, 1919), Hardy (1918, 1920), Kac and Wakimoto (1994), and many others.