Spectral methods for high-intensity ultrasound in complex geometries

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The Fourier continuation method for PDEs

We have developed a new numerical solver for systems of PDEs. The solver is based on the Fourier continuation (FC) method described in (Bruno and Lyon, 2009). The key qualities of the solver are

- Fast (FFT speed)
- High-order accuracy
- Small dispersion errors
- Applicable to general domains
- Parallel implementation


Continuation avoids Gibbs phenomenon

Problem: Fourier collocation methods suffer from the Gibbs phenomenon if the function is not periodic.

Solution: Continue the function into a slightly longer smooth periodic function.

The continuation is fast. On a 3.4GHz Intel CPU 1 million continuations in 0.85 seconds

High-order accuracy, small dispersion errors

The FC method approximates derivatives to about 5th order.

To the left we show the error in approximating the function

\[ f(x) = e^{\cos(2\pi x)}, \quad x \in [0, 1.1] \]

The FC solver applied to HIFU

As one application of the FC PDEs solver, we simulate high-intensity focused ultrasound (HIFU) waves by the Navier-Stokes equations. The waves are focused into the domain by an arc-shaped transducer. Note the effects of nonlinearity compared with the linear wave equation solution.

Here is the result when a scatterer is added.

For these simulations, we used a spatial step size \( h=1/40 \) (smaller near the cylinder). By comparison with the result of an \( h=1/60 \) simulation, we approximate the maximum error in the images above to be around 0.0006.

Conclusion

The Fourier continuation method leads to a fast and accurate method for solving HIFU problems in complex geometries. The spectral nature of the method allows us to perform direct numerical simulations of HIFU in large, complex domains with high-order accuracy and small dispersion errors.