On the mixed-type boundary condition for diffusion of heat on Cartesian grids using the immersed boundary method

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ABSTRACT

This note describes the implementation of a novel interpolation scheme which allows the accurate imposition of the mixed Dirichlet-Neumann boundary condition on curved geometries on Cartesian grids. The methodology is derived within the context of the immersed boundary technique. The method is general in the sense that the interpolation scheme does not involve any special treatment to handle either Dirichlet, Neumann or mixed (Robin) boundary conditions. An analysis of the accuracy of the interpolation algorithm on the boundary is carried out for the case of heat conduction in an annulus. The results show that the method is second-order and are in agreement with analytical and numerical data.

NOMENCLATURE

\begin{itemize}
\item \(a, b, c\) Constants in linear boundary condition \([\text{W/m}^2\text{K}]\)
\item \(d_j\) Set of constants in Eq. (9)
\item \(D_t\) Diffusive operator for temperature
\item \(H\) Nondimensional energy forcing
\item \(h_f\) Heat transfer coefficient of fluid \([\text{W/m}^2\text{K}]\)
\item \(k_s\) Thermal conductivity of body \([\text{W/m K}]\)
\end{itemize}

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Characteristic length [m]

Euclidean norm

Unit vector normal to surface

Projections on x- and y-axis of unit vector $n$

Normalized radius

Dimensional radius [m]

Dimensional time [s]

Characteristic time [s]

Dimensional temperature [$^\circ$C]

Reference temperature [$^\circ$C]

Fluid temperature [$^\circ$C]

Coordinates based on node $(i, j)$

Normalized Cartesian coordinates [m]

Coordinates of immersed-boundary node

Nondimensional thermal diffusivity, interpolation factor

Dimensional thermal diffusivity [m$^2$/s]

Interpolation factor

Mesh size, difference operator

Increment

Maximum norm

$j$-th eigenvalue

Gradient operator

Laplacian operator

Nondimensional temperature

Outer-boundary nondimensional temperature

Initial nondimensional temperature

Nondimensional fluid temperature

Temperature on or inside the body in Eq. (4)

Temperature outside the body in energy forcing interpolation

Nondimensional time

Indices for the Cartesian coordinates

Index of time step
1. INTRODUCTION

In recent years the use of computational fluid dynamics (CFD) methodologies, particularly the immersed boundary (IB) technique, has received great attention in the simulation of geometrically complex problems on Cartesian grids. Characteristics such as simplicity in grid generation, savings in memory and CPU time, and straightforward parallelization provided by the IB method, makes it a highly efficient prediction tool that can be used in the analysis and design of engineering equipment.

Though different versions of the IB technique have been developed and applied to fluid flow and heat transfer studies [1, 2, 3, 4, 5, 6, 7, 8], all of them are built upon the same principle, i.e., the application of a “forcing term” in the discretized momentum and/or energy equations, such that the boundary conditions on the body are satisfied [1, 5, 6]. These boundary conditions can be Dirichlet to specify a velocity or temperature, Neumann to define a flux (either mass or heat), or a combination of the two (Robin); the latter condition being a common situation in cases where diffusion processes within a solid surrounded by a fluid occur. On the other hand, regardless of the type of boundary condition applied, in Cartesian meshes the body does not usually coincide with the grid points. Thus, interpolation schemes are needed to properly enforce these conditions on the body surface.

Interpolation schemes, within the context of the IB, have been developed and applied mainly to carry out fluid flow studies, e.g. [5, 6, 8, 9], and to a lesser extent, to solve the passive-scalar field [1, 2, 7]. In these studies, the schemes reported are for either Dirichlet or Neumann conditions only. Moreover, even though interpolation algorithms for Dirichlet conditions have achieved great success, the advance in those for Neumann conditions has been relative. For instance, Fadlun et al. [7] reported that the adiabatic condition on the immersed boundary was not accurately satisfied with their method, whereas the accuracy
of the scheme developed by Kim et al. [2] to satisfy the iso-flux (Neumann) boundary condition was not ensured. Pacheco et al. [1], on the other hand, modified the IB of Kim et al. [6] to study fluid flow and heat transfer phenomena using non-staggered Cartesian grids. The reported interpolation procedures for Dirichlet conditions were different from those of Neumann owing to their specific nature. A numerical integration method suitable for the treatment of heat transfer problems on irregular domains under general (Robin) conditions was presented by Barozzi et al. [10]. Their technique, based on the immersed interface methods of [11, 12, 13], uses a second-order expansion of the normal derivative and a five-point computational stencil.

In the present study, we extend the applicability of the IB method to analyze heat transfer processes under mixed Dirichlet-Neumann conditions using a single interpolation scheme. To this end, a brief overview of the method is given first followed by details of the general interpolation scheme. Subsequently, we apply the mixed-boundary-condition algorithm, in the context of the IB, to a geometrically complex heat conduction problem to demonstrate the robustness of the approach. The method and boundary conditions are second order accurate as demonstrated by favorable comparison with analytical solutions and numerical results from a different technique.

2. IMMERSED BOUNDARY METHOD

To illustrate the idea of the IB technique, we consider the unsteady heat transfer by diffusion within a solid in a general domain \( D \) of boundary \( \partial D \), being this surrounded by a fluid. Its extension to simulations of heat convection phenomena (or any passive scalar-transport equation) is straightforward [1]. The nondimensional version of the governing
equations with constant properties are

\[ \frac{\partial \Theta}{\partial \tau} = \alpha \nabla^2 \Theta + H \quad \text{in} \quad D \]  
\[ a \Theta + b \nabla \Theta \cdot \mathbf{n} = c \quad \text{on} \quad \partial D \]  

where the nondimensional time \( \tau \), temperature within the solid \( \Theta \), temperatures Laplacian and gradient, \( \nabla^2 \Theta \) and \( \nabla \Theta \) respectively, and thermal diffusivity \( \alpha \) of the solid, are defined as

\[ \tau = \frac{\hat{t}}{t_c}, \quad \Theta = \frac{\hat{T} - \hat{T}_0}{T_\infty - \hat{T}_0}, \quad \nabla^2 \Theta = \frac{\nabla^2 \hat{T}}{L_c^2/(T_\infty - \hat{T}_0)}, \quad \nabla \Theta = \frac{\nabla \hat{T}}{L_c/(T_\infty - \hat{T}_0)}, \quad \alpha = \frac{\hat{\alpha}}{L_c^2/t_c}, \]

with \( H \) being the corresponding nondimensional energy forcing. In the above equation, the “hats” denote dimensional quantities. The constants in Eq. (2), defined as \( a = h_f \), \( b = k_s/L_c \), and \( c = h_f \Theta_\infty (\Theta_\infty = 1) \), all in \([W/m^2K]\), are the coefficients for the most general linear-boundary condition, and \( \mathbf{n} \) is the normal unit-vector. In the context of Cartesian grids, the Laplacian operator is \( \nabla^2 = \partial^2/\partial x_k \partial x_k \), and \( x_k = \hat{x}_k/L_c \), for \( j = 1, 2 \) in two dimensions. In the above formulation, \( \hat{T}_0 \) is a reference temperature, \( \hat{T}_\infty \) is the fluid temperature, \( L_c \) and \( t_c \) are the characteristic length and time, respectively, and \( h_f \) and \( k_s \) are the heat transfer coefficient of the fluid and thermal conductivity of the solid, respectively.

The discretized form of Eq. (1) can be written as

\[ \frac{\Theta^{n+1} - \Theta^n}{\Delta \tau} = \frac{1}{2} D_I (\Theta^{n+1} + \Theta^n) + H^{n+1/2}, \]  

where \( D_I = \delta^2/\delta x_k \delta x_k \) is the discrete Laplace operator, and \( \Delta \tau \) is the time-increment. In the context of the direct forcing method [8], \( H \) becomes

\[ H^{n+1/2} = \left( I - \frac{\Delta \tau}{2} D_I \right) \frac{\Theta - \Theta^n}{\Delta \tau} - D_I (\Theta^n), \]  

5
where \( \Theta \) refers to the temperature at the energy-forcing location that will ensure that the desired boundary condition is satisfied. When the location of \( H \) coincides with the boundary, then \( \Theta = \Theta \), otherwise \( \Theta \), must be obtained by interpolation from the surrounding temperature values. The details of the numerical solution in the context of the IB method can be found in [1] and references therein. Thence, in this paper we concentrate in the implementation of interpolation scheme for the mixed boundary conditions, i.e., Eq. (2).

3. DIRICHLET-NEUMANN BOUNDARY CONDITIONS

3.1 Interpolation scheme

In order to demonstrate the idea of the interpolation scheme, let us assume a two-dimensional body. In general, we may have two different types of nodes according to location on the body, i.e., nodes labeled (a), and those named either (b) or (c), as depicted in Figure 1. Shown in the figure are the unit vectors defining the tangent plane to the surface at each node considered, namely \( n_a \), \( n_b \) and \( n_c \).

We consider first the case where three nodal values surrounding point \((p, q)\) are known, and these nodes lie outside the body, as shown in Figure 2(a). This case corresponds to the cell on the left-hand-side of Figure 1, where point \((p, q)\) of Figure 2(a) is labeled as point (a) in Figure 1. For clarity in this analysis “bars” and “tildes” denote, respectively, values of temperature inside and outside the body.

The idea is to determine \( \Theta_{i,j} \), which is needed to compute \( H \) in Eq. (4), such that Eq. (2) is satisfied at node \((p, q)\). This is done using a bilinear interpolation scheme, as shown in Figure 3, where \( \alpha = (X - x_{i,j})/(x_{i+1,j} - x_{i,j}) \), \( \beta = (Y - y_{i,j})/(y_{i+1,j} - y_{i,j}) \), and \( \Theta_{p,q} = \Theta(X,Y) \).

Since we also need to compute derivatives, we require to have the values of auxiliary nodes, namely \( \Theta_{p,j}, \Theta_{p,j+1}, \Theta_{i,q} \) and \( \Theta_{i+1,q} \). Thus, we apply a linear interpolation scheme
on these nodes, as

\[
\tilde{\Theta}_{p,j} = \alpha \tilde{\Theta}_{i+1,j} + (1 - \alpha) \tilde{\Theta}_{i,j}, \tag{5a}
\]

\[
\tilde{\Theta}_{p,j+1} = \alpha \tilde{\Theta}_{i+1,j+1} + (1 - \alpha) \tilde{\Theta}_{i,j+1}, \tag{5b}
\]

\[
\tilde{\Theta}_{i,q} = \beta \tilde{\Theta}_{i,j+1} + (1 - \beta) \tilde{\Theta}_{i,j}, \tag{5c}
\]

\[
\tilde{\Theta}_{i+1,q} = \beta \tilde{\Theta}_{i+1,j+1} + (1 - \beta) \tilde{\Theta}_{i+1,j}, \tag{5d}
\]

and a bilinear interpolation scheme on \( \tilde{\Theta}_{p,q} \), as

\[
\tilde{\Theta}_{p,q} = \alpha \beta \tilde{\Theta}_{i+1,j+1} + \alpha (1 - \beta) \tilde{\Theta}_{i+1,j} + (1 - \alpha) \beta \tilde{\Theta}_{i,j+1} + (1 - \alpha)(1 - \beta) \tilde{\Theta}_{i,j}. \tag{6}
\]

On combining Eq. (2) with Eqs. (5a)-(5d) and Eq. (6), the value for \( \tilde{\Theta}_{i,j} \) can now be written as

\[
\tilde{\Theta}_{i,j} = \left( c - \alpha \beta a \tilde{\Theta}_{i+1,j+1} - \left[ \frac{\alpha b}{\delta y_j} n_y + \alpha(1 - \beta)a \right] \tilde{\Theta}_{i+1,j} \\
\quad - \left[ \frac{\beta b}{\delta x_i} n_x + (1 - \alpha) \beta a \right] \tilde{\Theta}_{i,j+1} - \frac{b}{\delta x_i} n_x \tilde{\Theta}_{i+1,q} - \frac{b}{\delta y_j} n_y \tilde{\Theta}_{p,j+1} \right) \\
\quad - \left[ \frac{(1 - \beta)b}{\delta x_i} n_x + \frac{(1 - \alpha)b}{\delta y_j} n_y + (1 - \alpha)(1 - \beta)a \right], \tag{7}
\]

where \( n_x \) and \( n_y \) are the projections of \( \mathbf{n} \) on the \( x \)- and \( y \)-axis, respectively. Note that Eq. (7) can be used to obtain \( \tilde{\Theta}_{i,j} \) under Dirichlet or Neumann conditions by setting either \( b = 0 \) or \( a = 0 \) in the equation.

Let us now consider the case where two nodal values surrounding point \((p, q)\) lie inside the body, as shown in Figure 2(b). This case corresponds to the cell on the center of Figure 1, where point \((p, q)\) of Figure 2(b) is labeled as point \((b)\) in Figure 1. Apparently, there are only two known temperatures outside the boundary \( (\tilde{\Theta}_{3b}, \tilde{\Theta}_{4b}) \) and two unknown temperatures inside the boundary \( (\tilde{\Theta}_{3b} \text{ and } \tilde{\Theta}_{2b}) \). However, we regard \( \tilde{\Theta}_{2b} \) as a known quantity, since \( \tilde{\Theta}_{2b} = \tilde{\Theta}_{1a} \), where \( \tilde{\Theta}_{1a} \) was previously obtained from Eq. (7) as
explained above. Therefore, we determine $\theta_{i,j}(=\Theta_{1b}$ of Figure 1) using Eq. (7) and the known surrounding values. Clearly, Eq. (7) can be used to determine the temperature values inside the body such that the desired boundary condition is satisfied. The solution procedure involves the following steps:

1. Find a nodal point where we want to satisfy the Robin boundary condition and three nodal points lie outside the body, e.g. node (a) of Figure 1.

2. Determine temperature $\theta_{i,j}(=\Theta_{1a})$ from the known surrounding values using Eq. 7.

3. Determine the corresponding node-temperature of the adjacent cell, e.g., node (b) of Figure 1, from Eq. (7) with $\tilde{\theta}_{i,j+1}$ being replaced by $\tilde{\theta}_{i,j+1}$. In this case $\tilde{\theta}_{i,j+1} = \tilde{\theta}_{1a}$ was previously determined from a bilinear interpolation along with the adjacent nodes external to the body $\tilde{\theta}_{2a}, \tilde{\theta}_{3a}, \tilde{\theta}_{4a}$, and Eq. (2) being evaluated at node (a), which are all known.

4. Repeat step 3 on the adjacent cell (right-hand-side of Figure 1).

5. Since $\tilde{\theta}_{1b}$ must equal $\tilde{\theta}_{2c}$, this procedure must be repeated until all the nodes near the body have been exhausted, and the difference in values between consecutive iterations is negligible, e.g., $\tilde{\theta}_{1b} - \tilde{\theta}_{2c} \approx 0$.

It is to be noted that, by slightly modifying this interpolation scheme, the method proposed here can be applied to an interface between two materials with different thermal conductivities. Studies on fluid-particle interaction (brownian motion of particles immersed in a fluid) with different thermal conductivities are currently under way, and the results will be reported elsewhere.

4. HEAT TRANSFER BY DIFFUSION IN AN ANNULUS

In order to test the correct implementation of the Robin formulation, along with the
numerical scheme, we carry out simulations of both steady-state and unsteady heat conduction in a two-dimensional annulus, shown schematically in Figure 4. As pointed out by Barozzi et al. [10] who used it to test an immersed interface (II) method, though relatively simple from an analytical perspective if one uses a cylindrical system, this problem is nontrivial in the frame of numerical Cartesian-grid approaches.

In reference to Figure 4, the nondimensional governing equations along with the initial and boundary conditions are

\[
\frac{\partial \Theta}{\partial \tau} = \alpha \nabla^2 \Theta, \tag{8a}
\]
\[
\Theta|_{\tau=0} = \Theta_0 \quad 1 \leq r \leq \frac{r_e}{r_i}, \tag{8b}
\]
\[
\Theta|_{\tau=\frac{r-e}{r_i}} = \Theta_e \quad \tau > 0, \tag{8c}
\]
\[
(a\Theta + b\nabla \Theta \cdot \mathbf{n})|_{r=1} = c \quad \tau > 0, \tag{8d}
\]

where the quantities in Eqs. (8a)–(8d) are defined as those in Eqs. (1)–(2), with \( \{x, y, r\} = \{\hat{x}, \hat{y}, \hat{r}\}/L_c \) and \( L_c = r_i \). The \( \nabla^2 \)- and \( \nabla \)-operators are applied accordingly with the chosen coordinate system, either Cartesian for the numerical technique or cylindrical for the analytical solution. Note that a constant temperature \( \Theta_e = 0 \) is applied at the external boundary whereas general conditions are applied at the internal one. Also, the initial temperature distribution is \( \Theta_0 = 0 \). For comparison purposes with the solutions obtained by Barozzi et al. [10] for the same problem, we use \( r_e = 2r_i \) such that \( r \in [1, 2] \).

### 4.1 Steady-state simulations

The computations were carried out for three cases in the internal boundary: (a) Dirichlet \( a = 1, \ b = 0, \ c = 1 \), (b) Neumann \( a = 0, \ b = 1, \ c = 1 \), and Robin \( a = 1, \ b = 1, \ c = 1 \) conditions. In all of the above, different number of regular grid points, ranging from \( 20 \times 20 \) to \( 120 \times 120 \), were used to assess the spatial accuracy of the methodology by comparison with both existent analytical solutions and the numerical results of Barozzi et al. [10].
For the three boundary conditions, errors in the solutions are presented in Figure 5 as functions of the mesh refinement $\delta$, in terms of the maximum ($\epsilon_{\text{max}}$)- and $L_2$-norms, with Figures 5(a), 5(b) and 5(c) showing the corresponding results for the Dirichlet, Neumann and Robin conditions, respectively. Also included in these are the results reported by Barozzi et al. [10]. From the figures it can be seen that both techniques achieve second-order accuracy, with actual numerical orders being quantitatively similar, as shown in Table 1. On the other hand, it can also be noticed that, regardless of the norm applied, the present formulation provides lower errors in the solution with coarser grids as compared to those of Barozzi et al. [10].

It is important to recognize that, despite the fact that the current implementation of the Robin boundary condition provides a second-order Cartesian scheme which is better than the II method of Barozzi et al. [10], as shown in Figure 5(c) there are uneven distributions of the errors in the solution as the mesh is refined. This was also noticed by Barozzi et al. [10], indicating that even in cases where satisfactory average accuracy are attained, high local errors could persist. A possible explanation is that due to the presence of irregular boundaries, the truncation error tends to increase locally; i.e., as the mesh is refined, the boundary might get closer to two grid points within the cell, thus causing a local increase in the error when computing the derivative. The same behavior in the error distribution can be observed in Figure 5(b) for the Neumann conditions for which the derivative is also needed.

### 4.2 Time-dependent simulations

Though the time accuracy of IB technique has already been assessed in [1], we illustrate its application in conjunction with the Robin implementation by solving the time-dependent system given in Eqs. (8a)–(8d), with $\alpha = 1$. 

On following [14], our solution of Eqs. (8a)-(8d) with \( \Theta_0 = 0, \Theta_e = 0, \) and \( a = b = c = 1 \) in Eq. (8d), is given in terms of Bessel series expansions as

\[
\Theta(r, \tau) = \sum_{j=1}^{\infty} d_j \left[ \frac{J_0(\lambda_j r)}{J_0(2\lambda_j)} - \frac{Y_0(\lambda_j r)}{Y_0(2\lambda_j)} \right] \exp(-\alpha \lambda_j^2 \tau) + \frac{\ln(r) + 1}{\ln(1/2) - 1},
\]

where the eigenvalues, \( \lambda_j \) for \( j = 1, 2, \cdots \), are the roots of the equation

\[
\frac{\lambda_j J_1(\lambda_j) + J_0(\lambda_j)}{J_0(2\lambda_j)} - \frac{\lambda_j Y_1(\lambda_j) + Y_0(\lambda_j)}{Y_0(2\lambda_j)} = 0,
\]

and the constants \( d_j \) for \( j = 1, 2, \cdots \), are defined as

\[
d_j = \frac{2 \lambda_j}{4 \left[ \frac{J_1(2\lambda_j)}{J_0(2\lambda_j)} - \frac{Y_1(2\lambda_j)}{Y_0(2\lambda_j)} \right]^2 - \left[ \frac{1}{\lambda_j^2} + 1 \right] \left[ \frac{J_0(\lambda_j)}{J_0(2\lambda_j)} - \frac{Y_0(\lambda_j)}{Y_0(2\lambda_j)} \right]^2}.
\]

In the above equations \( J_0, Y_0, J_1, \) and \( Y_1 \) are the Bessel functions of first and second kind, of order 0 or 1, respectively.

The corresponding time-dependent results for the temperature within the solid annulus are shown in Figure 6. Figure 6(a) illustrates the temperature distributions at values of the nondimensional time \( \tau = 0.15, 0.25, 0.35 \) and 1.40, whereas Figure 6(b) depicts the temporal evolution of the temperature at three different radii-locations, namely \( P_1 (r = 1.03) \) close to the inner boundary, \( P_2 (r = 1.49) \) approximately at the middle plane, and \( P_3 (r = 1.94) \) close to the outer boundary, as illustrated in Figure 4. From both figures it can be observed an excellent agreement between the analytical solutions (symbols) with those obtained numerically (solid lines). After \( \tau = 1.40 \) nondimensional units, the steady state has been reached, with a value of \( \Theta = 0.41 \) at the inner boundary and \( \Theta = 0.40 \) at \( P_1 \). It is to be noticed that these results diverge quantitatively from those of Barozzi et al. [10] (steady-state temperature of \( \Theta = 0.92 \) at approximately the same radius location), for this problem. On looking at the Robin condition imposed at the inner boundary, if we let the heat transfer coefficient \( h_f \) become sufficiently large, which is equivalent to let \( b \to 0, \)
then the mixed condition transforms into a Dirichlet one and $\Theta \to 1$ at the inner-radius location. Under such consideration, our results compare within 1% with those reported by Barozzi et al. [10].

\section{CONCLUSIONS}

In the simulation of geometrically complex phenomena, the immersed boundary method (IB) has become an attractive CFD technique due to its efficient mesh-generating process and savings in CPU time during calculations. However, very often the geometry of the body does not conform with the Cartesian-grid arrangement of the method and interpolation schemes are required to enforce the boundary conditions on the body surface.

In this work, we have presented in detail the construction of a general interpolation scheme that is able to handle the mixed Neumann-Dirichlet boundary conditions in the context of the IB technique. An advantage of this algorithm is that both Dirichlet and Neumann conditions can be naturally implemented. Its validation has been assessed by favorable comparison with numerical results available in the literature and analytical solutions for the steady- and unsteady-diffusion of heat in an annulus. The interpolation algorithm developed here provides a method that is second-order accurate in space and time, and its potential applications to study stratified flows, or mass transport, among other areas, is straightforward.

\section{ACKNOWLEDGMENTS}

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References


FIGURE CAPTIONS

1. Schematic for interpolation scheme at nodes (a), and (b) or (c).

2. Interpolation scheme for the Robin boundary condition. (a) Three nodes outside the boundary; (b) Two nodes outside the boundary.


4. Concentric cylindrical annulus in two-dimensions.

5. $L_2$-norm and maximum-norm ($\epsilon_{\text{max}}$) as functions of the mesh size $\delta$ for different boundary conditions. (a) Dirichlet ($a = 1, b = 0, c = 1$); (b) Neumann ($a = 0, b = 1, c = 1$); (c) Robin ($a = 1, b = 1, c = 1$).

6. Numerical (solid lines) vs. analytical (symbols) time-dependent solutions for the annulus problem. (a) Temperature distributions for various times; (b) Temperature vs. time for three radii-locations.

(a) $-\triangle - \tau = 0.15; - \circ - \tau = 0.25; - \sigma - \tau = 0.35; -\nabla - \tau = 1.40$.  
(b) $- \circ - r = 1.03; - \circ - r = 1.49; -\triangle - r = 1.94$. 

TABLE CAPTIONS

1. Numerical order of accuracy in terms of $L_2$ and $\epsilon_{\text{max}}$ for different types of implemented-boundary-conditions in the annulus problem.
Table 1: Numerical order of accuracy in terms of $L_2$ and $\epsilon_{\max}$ for different types of implemented-boundary-conditions in the annulus problem.

<table>
<thead>
<tr>
<th>Method / BC</th>
<th>Dirichlet</th>
<th>Neumann</th>
<th>Robin</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$L_2$</td>
<td>$\epsilon_{\max}$</td>
<td>$L_2$</td>
</tr>
<tr>
<td>Barozzi et al. [10]</td>
<td>1.91</td>
<td>1.93</td>
<td>2.05</td>
</tr>
<tr>
<td>Present</td>
<td>2.16</td>
<td>2.09</td>
<td>2.02</td>
</tr>
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</table>
Figure 1: Schematic for interpolation scheme at nodes (a), and (b) or (c).
Figure 2: Interpolation scheme for the Robin boundary condition. (a) Three nodes outside the boundary; (b) Two nodes outside the boundary.
Figure 3: Bilinear interpolation.
Figure 4: Concentric cylindrical annulus in two-dimensions.
Figure 5: $L_2$-norm and maximum-norm ($\epsilon_{\text{max}}$) as functions of the mesh size $\delta$ for different boundary conditions. (a) Dirichlet ($a = 1, b = 0, c = 1$); (b) Neumann ($a = 0, b = 1, c = 1$); (c) Robin ($a = 1, b = 1, c = 1$).
Figure 6: Numerical (solid lines) vs. analytical (symbols) time-dependent solutions for the annulus problem. (a) Temperature distributions for various times; (b) Temperature vs. time for three radii-locations.

(a) $-\Delta-\tau = 0.15; -\circ-\tau = 0.25; -\circ-\tau = 0.35; -\triangle-\tau = 1.40.$

(b) $-\circ- r = 1.03; -\circ- r = 1.49; -\triangle- r = 1.94.$