MAT 271 – Calculus II
Conic Sections

Here we give geometric definitions of parabolas, ellipses and hyperbolas and derive their standard equations.

1 Conic sections in Cartesian coordinates

- **Parabola**: Is the set of points in a plane that are equidistant from a fixed point \( F \) (called the *focus*) and a fixed line (called the *directrix*).
- **Vertex** is the point halfway between the focus and the directrix lying on the parabola.
- **Axis** of the parabola is the line through the focus perpendicular to the directrix.

![Figure 1: Parabola \( x^2 = 4py, \ p > 0 \).](image)

Let us place the vertex at the origin \( O \) and its directrix parallel to the \( x \)-axis. The focus is the point \((0,p)\), then the directrix has the equation \( y = -p \). If \( P(x, y) \) is any point on the parabola, then the distance from \( P \) to the focus is

\[
|PF| = \sqrt{x^2 + (y - p)^2}
\]

and the distance from \( P \) to the directrix is \( |y + p| \).

The defining property of the parabola is that these distances are equal:

\[
\sqrt{x^2 + (y - p)^2} = |y + p| \tag{2}
\]

An equivalent expression is obtained by squaring and simplifying:

- The equation of the parabola with focus \((0,p)\) and directrix \( y = -p \) is:

\[
x^2 = 4py \tag{3}
\]

- It opens upward if \( p > 0 \).
- It opens downward if \( p < 0 \).
Figure 2: (a) $x^2 = 4py$, $p > 0$. (b) $x^2 = 4py$, $p < 0$.

- The equation of the parabola with focus $(p, 0)$ and directrix $x = -p$ is:
  \[ y^2 = 4px \]  
  (4)

- It opens to the right if $p > 0$.
- It opens to the left if $p < 0$.

Figure 3: (a) $y^2 = 4px$, $p > 0$. (b) $y^2 = 4px$, $p < 0$.

- **Ellipse**: Is the set of points in a plane the sum of whose distances from two fixed points $F_1$ and $F_2$ is a constant. The two fixed points $F_1$ and $F_2$ are called **foci**.

Let us place the foci on the $x$-axis at the points $(-c,0)$ and $(c,0)$ so that the origin is halfway between the foci. Let the sum of the distances from a point on the ellipse to the foci be $2a > 0$. Then the point $P(x, y)$ is a point on the ellipse when

\[ |PF_1| + |PF_2| = 2a \]  
(5)
that is
\[ \sqrt{(x - c)^2 + y^2} + \sqrt{(x + c)^2 + y^2} = 2a \] (6)

which simplifies to
\[ (a^2 - c^2)x^2 + a^2y^2 = a^2(a^2 - c^2) \] (7)

For convenience, let \( b^2 = a^2 - c^2 \), then the equations of the ellipse becomes
\[ \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \] (8)

The ellipse
\[ \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad a \geq b > 0 \] (9)

has foci \((\pm c, 0)\) where \( c^2 = a^2 - b^2 \), and vertices \((\pm a, 0)\).

Figure 4: Foci \((\pm c, 0)\).

Figure 5: Foci \((0, \pm c)\).
The ellipse
\[ \frac{x^2}{b^2} + \frac{y^2}{a^2} = 1 \quad a \geq b > 0 \] (10)
has foci \((0, \pm c)\) where \(c^2 = a^2 - b^2\), and vertices \((0, \pm a)\).

- **Hyperbola**: Is the set of points in a plane the difference of whose distances from two fixed points \(F_1\) and \(F_2\) is a constant. The two fixed points \(F_1\) and \(F_2\) are called foci.

Let us place the foci on the \(x\)-axis at the points \((-c, 0)\) and \((c, 0)\) so that the origin is halfway between the foci. Let the difference of the distances from a point on the hyperbola to the foci be \(\pm 2a\). Then the point \(P(x, y)\) is a point on the hyperbola when
\[ |PF_1| - |PF_2| = \pm 2a \] (11)
that is
\[ \sqrt{(x - c)^2 + y^2} - \sqrt{(x + c)^2 + y^2} = 2a \] (12)
which simplifies to
\[ (a^2 + c^2)x^2 - a^2y^2 = a^2(a^2 + c^2) \] (13)
For convenience, let \(c^2 = a^2 + b^2\), then the equations of the ellipse becomes
\[ \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \] (14)

![Figure 6: (a) Foci \((\pm c, 0)\), (b) Foci \((0, \pm c)\).](image)

The hyperbola
\[ \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \quad a \geq b > 0 \] (15)
has foci \((\pm c, 0)\) where \(c^2 = a^2 + b^2\), and vertices \((\pm a, 0)\), and asymptotes \(y = \pm (b/a)x\).

The hyperbola
\[ \frac{y^2}{a^2} - \frac{x^2}{b^2} = 1 \quad a \geq b > 0 \] (16)
has foci \((0, \pm c)\) where \(c^2 = a^2 - b^2\), and vertices \((0, \pm a)\), and asymptotes \(y = \pm (a/b)x\).
2 Conic sections in Polar coordinates

Let \( F \) be a fixed point called focus and \( l \) be a fixed line called the directrix in a plane. Let \( e \) be a fixed positive number called eccentricity. The set of all points \( P \) in the plane such that

\[
\frac{PF}{Pl} = e
\]

(that is the ratio of the distance from \( F \) to the distance from \( l \) is the constant \( e \)) is a conic section. The conic is

1. an ellipse is \( e < 1 \)
2. a parabola is \( e = 1 \)
3. a hyperbola is \( e > 1 \)

Place the focus \( F \) at the origin and the directrix parallel to the \( y \)-axis and \( d \) units to the right. Thus, the directrix has equation \( x = d \) and is perpendicular to the polar axis. If the point \( P \) has polar coordinates \((r, \theta)\), we have

\[
r = e(d - r \cos \theta)
\]

or

\[
x^2 + y^2 = e^2(d - x)^2
\]

After completing the square, we have

\[
\left(x + \frac{d^2d}{1 - e^2}\right)^2 + \frac{y^2}{1 - e^2} = \frac{e^2d^2}{(1 - e^2)^2}
\]

![Figure 7: \( r = \frac{6}{1 - \sin \theta} \)]

If \( e < 1 \) we have the ellipse, it is of the form:

\[
\frac{(x - h)^2}{a^2} + \frac{y^2}{b^2} = 1
\]

where

\[
h = -\frac{e^2d}{1 - e^2}, \quad a^2 = \frac{e^2d^2}{(1 - e^2)^2}, \quad b^2 = \frac{e^2d^2}{1 - e^2}
\]
where
\[ c^2 = a^2 - b^2 = \frac{d_1 d_2}{(1 - e^2)^2} \]

It also follows that the eccentricity is given by

\[ e = \frac{c}{a} \]  \hspace{1cm} (22)

If \( e > 1 \) we have the hyperbola, it is of the form:

\[ \frac{(x - h)^2}{a^2} - \frac{y^2}{b^2} = 1 \]  \hspace{1cm} (23)

and see that

\[ e = \frac{c}{a} \quad c^2 = a^2 + b^2 \]

By solving (18) se see that the polar equation of the conic can be written as

\[ r = \frac{ed}{1 + e \cos \theta} \]  \hspace{1cm} (24)

A polar equation of the form

\[ r = \frac{ed}{1 \pm e \cos \theta} \quad r = \frac{ed}{1 \pm e \sin \theta} \]

represents a conic section with eccentricity \( e \). The conic is an ellipse if \( e < 1 \), a parabola if \( e = 1 \) or a hyperbola if \( e > 1 \).
Figure 9: $r = \frac{6}{1+2\sin\theta}$