Notes on Intensity to Rotational Velocity

Rosemary Renaut

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First Pass

The shift for time after maximum intensity is reached

First assume that the intensity $I(t)$ is normalized with respect to maximum value $I_{\text{max}}$, so the normalized value satisfies $0 \leq I(t) \leq 1$, this is needed to take note the relation between the rotation angle $\theta(t)$ and the intensity, $I(t)$. To allow the case that the relationship is sinusoidal but potentially phase shifted, we introduce constant phase shift $\phi$. If $\phi = 0$, then $\theta = 0$ when $I = 0$. Otherwise $\phi$ accounts for phase shift $\theta = -\phi$ when $I = 0$. Let $t_0$ be such that $I(t_0) = 0$ is the first time at which the intensity is zero, then $\theta(t_0) = -\phi(t_0)$. Moreover, let $t_{\text{max}}$ be the first time at which $I(t) = I_{\text{max}}$, i.e. $I$ is increasing for $t_0 < t < t_{\text{max}}$, with $0 < I(t) < 1$. We assume the relationship

$$\sin(\theta + \phi) = I(t)$$

thus

$$\theta(t) = \arcsin(I(t)) - \phi = \arcsin(I(t)) + \theta(t_0), \quad t_0 < t < t_{\text{max}}. \quad (1)$$

1. Note that we can evaluate $\theta$ in radians using the matlab function $\text{asin}$ and in degrees using the function $\text{asind}$, and thus provided that $\theta(t_0)$ is measured consistently in either radians or degrees, we do not need to express the relationships with the scalings between radians and degrees.

2. The relation (1) is valid for $-\pi/2 < \theta + \phi < \pi/2$. We assume $0 < \theta + \phi < \pi/2$ for the physical set up, i.e. measuring only positive angles.

3. For $\pi/2 < \theta + \phi < \pi$, we have to shift (1)

$$\sin(\theta + \phi) = I(t), \quad t > t_{\text{max}} \quad (2)$$

implies

$$\theta(t) + \phi = \pi - \arcsin(I(t)) \quad (3)$$

$$\theta(t) = \pi - \arcsin(I(t)) + \theta(t_0), \quad t > t_{\text{max}}. \quad (4)$$

**Derivatives**

Now we use the chain rule on (1)

$$\sin(\theta + \phi) = I(t)$$

$$\cos(\theta + \phi) \frac{d\theta}{dt} = \frac{dI(t)}{dt} \quad \text{implies when } \theta + \phi \neq \pi/2$$

$$\frac{d\theta}{dt} = \frac{1}{\cos(\theta + \phi)} \frac{dI(t)}{dt} = \frac{1}{\sqrt{1 - \sin^2(\theta + \phi)}} \frac{dI(t)}{dt}$$

$$\frac{d\theta}{dt} = \frac{1}{\sqrt{1 - I^2(t)}} \frac{dI(t)}{dt} \quad t_0 < t < t_{\text{max}} \quad \text{and} \quad t > t_{\text{max}}. \quad (5)$$

Hence rather than using differencing on $\theta$ one should obtain the derivative from differencing on $I(t)$.

There is still the question at $\cos(\theta + \phi) = 0$. But $\cos(\theta + \phi) = 0$ corresponds to the maximum of the intensity, so $\frac{dI(t)}{dt} = 0$, which is consistent. To obtain the derivative for $\theta$ at this argument
θ + φ = π/2 we take the derivative again
\[ \frac{d}{dt} \left( \cos(\theta + \phi) \frac{d\theta}{dt} \right) = \cos(\theta + \phi) \frac{d^2\theta}{dt^2} - \sin(\theta + \phi) \left( \frac{d\theta}{dt} \right)^2 = \frac{d^2 I}{dt^2} \]
implies \[ \left( \frac{d\theta}{dt} \right)^2 = -\frac{d^2 I}{dt^2} > 0, \] (6)
where the last inequality follows because \( \theta + \phi = \pi/2 \) is the point at which \( I \) is maximum, so its second derivative is negative. Thus to evaluate the derivative at the maximum for the intensity you need the second derivative for \( I \)

\[ \frac{d\theta}{dt} = \sqrt{-\frac{d^2 I}{dt^2}}, \quad t = t_{\text{max}}. \] (7)

Also notice that in the neighborhood of \( I(t) = 1 \) there is also a difficulty with evaluating the derivative. Let \( \theta(t) + \phi = \pi/2 - \epsilon \) where \( \epsilon << 1 \). Then
\[ \cos(\theta(t) + \phi) = \sin(\epsilon) \quad \sin(\theta(t) + \phi) = \cos(\epsilon) \]
and we use again the second order derivatives to approximate
\[ \frac{d\theta}{dt} = \sqrt{\frac{1}{\cos(\epsilon)}} \frac{d^2 I}{dt^2}, \quad \theta(t) + \phi = \pi/2 - \epsilon \] (8)
which also holds for \( \theta(t) + \phi = \pi/2 + \epsilon \). Rather than expressing in terms of the estimate angle we can also write this directly using the intensity. Let \( I(t) = 1 - \zeta \) where \( 0 < \zeta << 1 \), then \((1 - I(t)^2) = (1 - (1 - 2\zeta + \zeta^2)) = 2\zeta(1 - \zeta/2) \) which yields \((1 - I(t)^2)^{-1/2} \approx (2\zeta)^{-1/2} \) and
\[ \frac{d\theta}{dt} = \frac{1}{\sqrt{1 - I(t)^2}} \frac{dI(t)}{dt} \]
\[ \approx \frac{1}{\sqrt{2\zeta}} \frac{dI(t)}{dt} \quad I(t) = 1 - \zeta. \] (9, 10)

Define a range for which one uses the approximation by \( \zeta_s = 1 - I(t_s) \) then defining the local \( \zeta \)
\[ \frac{d\theta}{dt} \approx \frac{1}{\sqrt{2\zeta}} \frac{dI(t)}{dt} \quad I(t) = 1 - \zeta, \quad t_s < t < t_{\text{max}}. \] (11)

The equations are given in radians, to find the derivative in degrees, let \( \delta(t) \) be the angle in degrees. \( \delta(t) = \theta(t) \frac{180}{\pi} \). Then
\[ \frac{d\delta}{dt} = \frac{180}{\pi} \frac{d\theta}{dt}. \] (12)
In summary, using (11), and noting $I(t_{\text{max}} + sR) = 1 - \zeta$

$$\delta(t) = \frac{180}{\pi} \begin{cases} \arcsin(I(t)) + \theta(t_0) & t_0 < t < t_{\text{max}} \\ \frac{\pi}{2} + \theta(t_0) & t = t_{\text{max}} \\ \pi - \arcsin(I(t)) + \theta(t_0), & t > t_{\text{max}}. \end{cases}$$

(13)

$$\frac{d\delta}{dt} = \frac{180}{\pi} \begin{cases} \frac{1}{\sqrt{1 - I^2(t)}} \frac{dI(t)}{dt} & t_0 < t < s \\ \frac{1}{2} \frac{d^2I(t)}{dt^2} & I(t) = 1 - \zeta, \ t_s < t < t_{\text{max}} \\ \sqrt{-\frac{d^2I(t)}{dt^2}} & t = t_{\text{max}} \\ \frac{1}{2} \frac{dI(t)}{dt} & I(t) = 1 - \zeta, \ t_{\text{max}} < t < t_{\text{max}} + sR \\ \frac{1}{\sqrt{1 - I^2(t)}} \frac{dI(t)}{dt} & t > t_{\text{max}} \end{cases}$$

(14)

and we can use the middle equation in the neighborhood of $t = t_{\text{max}}$.

Direct use of the intensity rather than the angles to evaluate the derivatives should stabilize the calculations, although one still needs to be careful at the maximum value for the intensity and use the limiting form (8).

**Notes on differencing**

To estimate the derivatives again use differencing on the intensity, as previously applied for the angle, e.g.

$$\frac{dI}{dt} \approx \frac{I(t_{k+1}) - I(t_k)}{t_{k+1} - t_k}$$

(15)

$$\frac{d^2I}{dt^2} \approx \frac{I(t_{k+1}) - 2I(t_k) + I(t_{k-1})}{(t_{k+1} - t_k)^2}$$

(16)

where $t_k$ is the time point for the measurements.

**Approximation**

Suppose that $I(t)$ is linear in time $I(t) = at + b$ for some $a$ and $b$, that could be found by regression, and for $0 < t \leq t_{\text{max}}$. $I(t)$ used in the estimates is normalized so that $I(t_{\text{max}}) = 1 = at_{\text{max}} + b$, thus $a = (1-b)/t_{\text{max}}$ implies $I(t) = (1-b)(t/t_{\text{max}}) + b$. This gives

$$\frac{dI}{dt} = \frac{1-b}{t_{\text{max}}}$$

(17)

$$\frac{d^2I}{dt^2} = 0$$

(18)

$$\sqrt{1 - I(t)^2} = \sqrt{1 - ((1-b)(t/t_{\text{max}}) + b)^2} = \frac{1}{t_{\text{max}}} \sqrt{t_{\text{max}}^2 - ((1-b)t + bt_{\text{max}})^2}$$

(19)

$$= \sqrt{(1-b)(t_{\text{max}} - t)(t_{\text{max}} + t + b(t_{\text{max}} - t)))}$$

(20)

$$\frac{d\delta}{dt} = \frac{180}{\pi} \begin{cases} \sqrt{\frac{1-b}{(t_{\text{max}} - t)(t_{\text{max}} + t + b(t_{\text{max}} - t))}} & t_0 < t < t_{\text{max}} \\ 0 & t = t_{\text{max}} \end{cases}$$

(21)
1. Notice if $I$ is determined such that $b = 0$, i.e. such that the linear approximation goes through $(0,0)$. Then we have

$$\frac{d\delta}{dt} = \frac{180}{\pi} \begin{cases} \frac{1}{\sqrt{t_{max}^2 - t^2}} & t_0 < t < t_{max} \\ 0 & t = t_{max} \end{cases},$$

(22)

2. If $I$ is such that the linear approximation holds for an interval to the left of $t_{max}$ we can work out $b$ by regression for the relevant interval. Then the slope of the rate of change of the angle is determined by the linear approximation. **Can we use this to find out something about different configurations?**