Ill-Posed Inverse Problems: Algorithms and Applications
Total Least Squares

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Inverse Problems

Total Least Squares Methods

Results

Conclusions

Other Research
Parameter Estimation Problem

- **Classical Approach** Linear Least Squares ($A$ is exactly specified.)

\[ x_{LS} = \arg \min_x ||Ax - b||^2 \]

- Orthogonal projection of $b$ onto the range of $A$. 
Parameter Estimation Problem

- **Classical Approach** Linear Least Squares ($A$ is exactly specified.)
  \[ x_{LS} = \arg\min_{x} ||Ax - b||^2 \]
  - Orthogonal projection of $b$ onto the range of $A$.

- **Dense** A Form $QR$ Decomposition of $A$, solve directly
  \[ Rx = Q^T b. \]
Parameter Estimation Problem

- Classical Approach Linear Least Squares ($A$ is exactly specified.)

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- Orthogonal projection of $b$ onto the range of $A$.

- Dense $A$ Form $QR$ Decomposition of $A$, solve directly

\[ Rx = Q^T b \]

- Sparse $A$ Use iterative techniques, CG, Krylov subspace, etc.
Integral Equations

\[ \int_{\Omega} \text{input} \times \text{system} \, d\Omega = \text{output} \]
Integral Equations

\[ \int_{\Omega} \text{input } X \text{ system } d\Omega = \text{output} \]

- Given noisy output determine input *or and* the system.
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- \textbf{General Applications:} Image Processing, Signal Identification
Integral Equations

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- Given noisy output determine input \textit{or/and} the system.
- \textbf{General Applications}: Image Processing, Signal Identification
- \textbf{Our Motivation} Seismic & Medical Signal Restoration
Signal degradation is modeled as a convolution

\[ g = f \otimes h + n \]

- where \( g \) is the blurred signal
- \( f \) is the unknown signal
- \( h \) is the point spread function (PSF) - known
- \( n \) is noise
Invariant Model

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- \( n \) is noise

- **Matrix Formulation** \( H \) is Toeplitz (structured)

\[ g = Hf + n \]
Example Of Convolution

\[ g = f \otimes h + n \]
Inverse with Known PSF

- Find $f$ from $g = f \otimes h + n$ given $g$ and $h$ with unknown $n$. 
Inverse with Known PSF

- Find $f$ from $g = f \otimes h + n$ given $g$ and $h$ with unknown $n$.
- Assuming normal distributed $n$ yields the estimator

$$\hat{f} = \arg \min_f \{ \| g - f \otimes h \|^2 \}$$
Inverse with Known PSF

- Find $f$ from $g = f \otimes h + n$ given $g$ and $h$ with unknown $n$.
- Assuming normal distributed $n$ yields the estimator
  $$\hat{f} = \arg \min_f \{ \| g - f \otimes h \|_2^2 \}$$
- Reconstruction with $n \in \mathbb{N}(0, 10^{-7})$
Ill-posedness

- Add more information about the signal
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- e.g. statistical properties or information about the structure
  (e.g. sparse decon, or total variation decon)
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- e.g. statistical properties or information about the structure (e.g. sparse decon, or total variation decon)
- Regularize

\[ \hat{f} = \arg \min_f \{ \| g - f \otimes h \|_2^2 + \lambda R(f) \} , \]

where \( R(f) \) is the penalty term and \( \lambda \) is a penalty parameter.
Standard Methods

- Tikhonov (TK).

\[ R(f) = \text{TK}(f) = \int_{\Omega} |\nabla f(x)|^2 dx. \]
Standard Methods

- **Tikhonov (TK)**
  \[ R(f) = TK(f) = \int_{\Omega} |\nabla f(x)|^2 dx. \]

- **Total Variation (TV)**
  \[ R(f) = TV(f) = \int_{\Omega} |\nabla f(x)| dx. \]
Standard Methods

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- Total Variation (TV)

\[ R(f) = TV(f) = \int_{\Omega} |\nabla f(x)| dx. \]

- Sparse deconvolution ($L^1$)

\[ R(f) = \|f\|_1 = \int_{\Omega} |f(x)| dx. \]
Comparing TV and TK Regularization

![Graph comparing TV and TK solutions](image-url)
\[ \hat{f} = \arg \min_{f} \{ \|g - f \otimes h\|_2^2 + \lambda R(f) \} \]

- \( \lambda \) Governs the trade off between the fit to the data and the smoothness of the reconstruction and can be picked by the L-curve approach.
Cost Functional

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- TV yields a piece wise constant reconstruction and preserves the edges of the signal.
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- TV yields a piece wise constant reconstruction and preserves the edges of the signal
- TK yields a smooth reconstruction
- To find the minimum we use a limited memory BFGS method
Optimization

- TK is a linear least squares (LS) problem

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The TV objective function is non differentiable

\[ J(f) = \| g - Hf \|_2^2 + \lambda \| Lf \|_1 \]
Optimization

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- Evaluation of the OF and its gradient is cheap (some FFTs and sparse matrix-vector multiplications)
Simulated PET

- Blur Segmented MRI scan using **Gaussian PSF**
Simulated PET

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- Add Gauss noise to resulting Simulated PET image

![Simulated PET Images]
Simulated PET

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**Note:** PSF is known regularize with TV
Real PET data

- Reconstruction done using Filtered Back Projection
- PSF estimated by a Gaussian
- TV regularization
Observations

- Image improvement is possible even with a rough estimation of the PSF (non-blind decon)
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- Image improvement is possible even with a rough estimation of the PSF (non-blind decon)
- Total Variation regularization (piecewise constant solution) is appropriate: intensity levels depend on the tissue type.
- Improvement requires better approximation of the PSF
- Increased Artifacts and noise. (More post processing can improve this)
Inverse Problems
  Linear Parameter Estimation
  Integral Equations
  Structured Inverse Problem
  Regularization
  PET Examples
  Unstructured Inverse Problems

Total Least Squares Methods

Results

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Other Research
Matrix Formulation: $H$ not Toeplitz

- Consider $g = Hf + n$ where either $H$ is structured but not known, or $H$ is estimated and unstructured.
Matrix Formulation: $H$ not Toeplitz

- Consider $g = Hf + n$ where either $H$ is structured but not known, or $H$ is estimated and unstructured.

- **Example: Dynamic PET** Estimate parameters $\mathcal{K} = (K_1, k_2, k_3, k_4)$, which satisfy

$$y(t) = u(t) \otimes \left( c_1(\mathcal{K})e^{-\lambda_1(\mathcal{K})t} + c_2(\mathcal{K})e^{-\lambda_2(\mathcal{K})t} \right).$$

Nonlinear parameter estimation can be converted to linear form

$$H = \left[ \int (u); \int \int (u); \int (y); \int \int \int (y) \right],$$

$g = Hf$, but $f = f(\mathcal{K})$, is non linear.
Inverse Problems

Total Least Squares Methods

History
Total Least Squares
Regularization
Solution Techniques
Theoretical Results
Algorithm

Results

Conclusions

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• **Statistics:** Errors in variables regression (Gleser, 1981)
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  - Measurement error models (Fuller, 1987)
  - Orthogonal distance regression (Adcock, 1878)
  - Generalized least squares (Madansky, 1959,...)
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  - eigenvector method (Levin, 1964)
  - Koopmans-Levin method (Fernando & Nicholson, 1985)
  - Compensated least squares (Stoica & Soderstrom, 1982)
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• **Signal Processing:** Minimum norm method (Kumaresan & Tufts, 1983)
\[ \min \| (E, f) \|_F \quad \text{subject to} \quad (A + E)x = b + f \]

- **Classical Algorithm** Golub Reinsch Van-Loan

\[
\begin{bmatrix} \hat{A} \\ \hat{b} \end{bmatrix} \begin{bmatrix} x^{TLS} \\ -1 \end{bmatrix} = 0, \quad \hat{A} = A + E, \quad \hat{b} = b + f. \]
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\min \|(E, f)\|_F \quad \text{subject to} \quad (A + E)x = b + f
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- **Classical Algorithm** Golub Reinsch Van-Loan
  
  Solve \([\hat{A} | \hat{b}]\) \[x^{TLS} \atop -1\] = 0, \(\hat{A} = A + E\), \(\hat{b} = b + f\).

- **SVD Solution** uses the SVD of augmented matrix \([A | b]\).

  \[
  \begin{bmatrix}
  x^{TLS} \\
  -1
  \end{bmatrix} = \frac{-1}{v_{n+1,n+1}} v_{n+1}.
  \]
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- **Direct** Compute the SVD and solve.
Rayleigh Quotient Formulation for TLS

- An Iterative Approach:

\[ x_{TLS} = \arg\min_x \phi(x) = \arg\min_x \frac{\|Ax - b\|^2}{1 + \|x\|^2}, \]
Rayleigh Quotient Formulation for TLS

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- \(\phi\) is the Rayleigh quotient of matrix \([A, b]\).
- TLS minimizes the sum of squared normalized residuals.
An eigenvalue problem for TLS

- Björck, Hesternes and Matstoms (BHM), 2000

Eigenvalue equation for TLS

\[
\begin{pmatrix}
  A^T A & A^T b \\
  b^T A & b^T b \\
\end{pmatrix}
\begin{pmatrix}
  x \\
  -1 \\
\end{pmatrix}
= \sigma^2 \begin{pmatrix}
  x \\
  -1 \\
\end{pmatrix}
\]

\(\sigma^2\) is to be minimized.

System matrix is constant.

First block equation gives the normal equations for TLS

\[
(A^T A - \sigma^2 I) x = A^T b.
\]

Use Rayleigh quotient iteration to find the minimal eigenvalue.

Solve shifted normal equations by Preconditioned Conjugate Gradients each iteration.
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- Solve \emph{shifted normal equations} by Preconditioned Conjugate Gradients each iteration.
Tikhonov TLS

**Regularization** Stabilize TLS with realistic constraint on data $\|Lx\| \leq \delta$. $\delta$ is prescribed from known information on the solution, and $L$ is typically a differential operator.
Tikhonov TLS

Regularization  Stabilize TLS with realistic constraint on data $\|Lx\| \leq \delta$. $\delta$ is prescribed from known information on the solution, and $L$ is typically a differential operator.

Reformulation with Lagrange Multiplier  If the constraint is active the solution $x^*$ satisfies *normal equations*

\[(A^TA + \lambda_I I + \lambda_L L^T L)x^* = A^T b, \quad (x^*)^T L^T L x^* - \delta^2 = 0,\]

\[\lambda_I = -\frac{\|Ax^* - b\|^2}{1 + \|x^*\|^2}, \quad \lambda_L = \mu (1 + \|x^*\|^2),\]

\[\mu = -\frac{1}{\delta^2} \left( \frac{b^T (Ax^* - b)}{1 + \|x^*\|^2} + \frac{\|Ax^* - b\|^2}{(1 + \|x^*\|^2)^2} \right),\]

Golub, Hansen and O’Leary, 1999. Solution technique is parameter dependent: $\lambda_L, \lambda_I, \delta, \mu$.
Rayleigh Quotient Formulation for RTLS

- Recall

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- Formulate regularization for TLS in RQ form

\[ \min_x \phi(x) \text{ subject to } \|Lx\| \leq \delta. \]
Rayleigh Quotient Formulation for RTLS

- Recall

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- Formulate *regularization for TLS in RQ form*

\[ \min_x \phi(x) \text{ subject to } \|Lx\| \leq \delta. \]

- Augmented Lagrangian

\[ \mathcal{L}(x, \mu) = \phi(x) + \mu(\|Lx\|^2 - \delta^2). \]
Eigenvalue Problem RTLS: Renaut & Guo (SIAM 2004)

\[
B(x^*) \begin{pmatrix} x^* \\ -1 \end{pmatrix} = -\lambda_l \begin{pmatrix} x^* \\ -1 \end{pmatrix},
\]

\[
B(x^*) = \begin{pmatrix} A^T A + \lambda_L(x^*)L^T L, & A^T b \\ b^T A, & -\lambda_L(x^*)\gamma + b^T b \end{pmatrix},
\]

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\lambda_L = \mu(1 + \|x^*\|^2),
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- Seek the minimal eigenpair for \( B \). Note \( B \) is not constant.
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\[ \lambda_L = \mu(1 + \|x^*\|^2), \]

- Seek the minimal eigenpair for \( B \). Note \( B \) is not constant.
- Specify constraint \( \|Lx^*\|^2 \leq \delta^2 \), use \( \gamma = \delta^2 \) or \( \|Lx^*\|^2 \)
• At a solution constraint is active: measure normalized discrepancy

\[ g(x) = \frac{\|Lx\|^2 - \delta^2}{1 + \|x\|^2} \]
Development

- At a solution constraint is active: measure normalized discrepancy
  \[ g(x) = \frac{\|Lx\|^2 - \delta^2}{1 + \|x\|^2} \]

- Denote \( B(\theta) = M + \theta N, \quad N = \begin{pmatrix} L^T L & 0 \\ 0 & -\delta^2 \end{pmatrix} \).
Development

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- Denote eigenpair for smallest eigenvalue of \( B(\theta) \) as \( \varrho_{n+1}, (x_\theta^T, -1)^T \).
Development

- At a solution constraint is active: measure normalized discrepancy
  \[ g(x) = \frac{(\|Lx\|^2 - \delta^2)}{(1 + \|x\|^2)} \]

- Denote \( B(\theta) = M + \theta N \), \( N = \begin{pmatrix} L^TL & 0 \\ 0 & -\delta^2 \end{pmatrix} \).

- Denote eigenpair for smallest eigenvalue of \( B(\theta) \) as \( \varrho_{n+1}, (x^T_\theta, -1)^T \)

- Reformulation:
  
  **For a constant \( \delta \), find a \( \theta \) such that \( g(x_\theta) = 0 \).**
Lemma Assuming that $b^TA \neq 0$ and $\mathcal{N}(A) \cap \mathcal{N}(L) = \{0\}$, then the smallest eigenvalue of $B(\theta)$ is simple.
Lemma Assuming that $b^T A \neq 0$ and $\mathcal{N}(A) \cap \mathcal{N}(L) = \{0\}$, then the smallest eigenvalue of $B(\theta)$ is simple.

Lemma If $[A, b]$ is a full rank matrix, there exists one and only one positive number, denoted by $\theta^c$, such that $B(\theta^c)$ is singular, and

(i) the null eigenvalue of $B(\theta^c)$ is simple
(ii) when $0 \leq \theta < \theta^c$, $B(\theta)$ is positive definite, and
(iii) when $\theta > \theta^c$, $B(\theta)$ has only one negative eigenvalue.
Lemma If $b^T A \neq 0$, $[A, b]$ is full-rank, then solution of $g(x_\theta) = 0$ is unique and

(i) There exists a $\lambda^*_L \in [0, \theta^c]$ which solves $g(x_\theta) = 0$.
(ii) When $\lambda_L \in (0, \lambda^*_L)$, $g(x_{\lambda_L}) > 0$ and $\lambda_L \in (\lambda^*_L, \infty)$, $g(x_{\lambda_L}) < 0$. 
Uniqueness

Lemma  If $b^T A \neq 0$, $[A, b]$ is full-rank, then solution of $g(x_\theta) = 0$ is unique and

(i) There exists a $\lambda^*_L \in [0, \theta^c]$ which solves $g(x_\theta) = 0$.

(ii) When $\lambda_L \in (0, \lambda^*_L)$, $g(x_{\lambda_L}) > 0$ and $\lambda_L \in (\lambda^*_L, \infty)$, $g(x_{\lambda_L}) < 0$.

Observe  We see from this result that there is an unique solution to our problem and that an algorithm for finding this solution should depend both on finding an update for the Lagrange parameter $\lambda_L$ and monitoring the sign of $g(x_{\lambda_L})$. 
The Update Equation for $\theta = \lambda_L$

$$
\lambda^{(k+1)}_L = \lambda^{(k)}_L + \iota^{(k)} \frac{\lambda^{(k)}_L}{\delta^2} g(x^{(k)}_L),
$$

$$
0 < \iota^{(k)} \leq 1 \text{ such that } g(x^{(k+1)}_L)g(x^{(0)}_L) > 0.
$$
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\[0 < \iota^{(k)} \leq 1\] such that \[g(x_{\lambda_L^{(k+1)}})g(x_{\lambda_L^{(0)}}) > 0.\]

Theorem **Given** $\lambda_L^{(0)} > 0$ **iteration converges to unique solution**, $\lambda_L^*$. 

*Rosemary Renaut with Dr Hongbin Guo and Wolfgang Stefan* 

**Ill-Posed Inverse Problems: Algorithms and Applications*
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$$0 < \iota^{(k)} \leq 1 \text{ such that } g(x_{\lambda_L^{(k+1)}}) g(x_{\lambda_L^{(0)}}) > 0.$$

Theorem **Given** $\lambda_L^{(0)} > 0$ **iteration converges to unique solution**, $\lambda_L^*$.

Observe $B(\lambda_L^{(k)})$ is always positive definite. For $0 < \lambda_L^{(0)} < \theta^c$ iteration is monotonically increasing or decreasing dependent on $\lambda_L^{(0)} >> \lambda_L^*$. 

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Ill-Posed Inverse Problems: Algorithms and Applications
EXACT RTLS: Alternating Iteration on $\lambda_L$ and $x$.

Algorithm

Given $\delta$, $\lambda^{(0)}_L > 0$, calculate eigenpair $(\varrho^{(0)}_{n+1}, x^{(0)})$. Set $k = 0$. Update $\lambda^{(k)}_L$ and $x^{(k)}$ until convergence.

1. While not converged Do $\iota^{(k)} = 1$
EXACT RTLS: Alternating Iteration on $\lambda_L$ and $x$.

Algorithm

Given $\delta$, $\lambda_L^{(0)} > 0$, calculate eigenpair $(\varrho_n^{(0)}, x^{(0)})$. Set $k = 0$. Update $\lambda_L^{(k)}$ and $x^{(k)}$ until convergence.

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1.1 Inner Iteration: Until $g(x^{(k+1)})g(x^{(0)}) > 0$
EXACT RTLS: Alternating Iteration on $\lambda_L$ and $x$.

Algorithm

Given $\delta$, $\lambda^{(0)}_L > 0$, calculate eigenpair $(\varrho^{(0)}_{n+1}, x^{(0)})$. Set $k = 0$. Update $\lambda^{(k)}_L$ and $x^{(k)}$ until convergence.

1. While not converged Do $\iota^{(k)} = 1$

1.1 Inner Iteration: Until $g(x^{(k+1)})g(x^{(0)}) > 0$

1.1.1 Update $\lambda^{(k+1)}_L$. 
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EXACT RTLS: Alternating Iteration on $\lambda_L$ and $x$.

**Algorithm**

*Given $\delta$, $\lambda_L^{(0)} > 0$, calculate eigenpair $(\varrho_{n+1}^{(0)}, x^{(0)})$. Set $k = 0$. Update $\lambda_L^{(k)}$ and $x^{(k)}$ until convergence.*

1. While not converged **Do** $\iota^{(k)} = 1$
   
   1.1 **Inner Iteration:** Until $g(x^{(k+1)})g(x^{(0)}) > 0$
      
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1.2 Test for convergence. If converged **Break** else $k = k + 1$. 
EXACT RTLS: Alternating Iteration on $\lambda_L$ and $x$.

Algorithm

Given $\delta$, $\lambda_L^{(0)} > 0$, calculate eigenpair $(\rho_{n+1}^{(0)}, x^{(0)})$. Set $k = 0$. Update $\lambda_L^{(k)}$ and $x^{(k)}$ until convergence.

1. While not converged Do $i^{(k)} = 1$
   
   1.1 **Inner Iteration:** Until $g(x^{(k+1)})g(x^{(0)}) > 0$
      
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2. End Do. $x_{RTLS} = x^{(k)}$. 
Inner Iteration Solve I

- Find eigenvector \( y^{(k,j+1)} = ((x^{(k,j+1)})^T, -1)^T \) such that

\[
B(\lambda_L^{(k)})y^{(k,j+1)} = \beta_{(k,j)}y^{(k,j)},
\]

\[
B(\lambda_L^{(k)}) = \begin{pmatrix}
J^{(k,j)} & A^T b \\
b^T A & \eta_{(k,j)}
\end{pmatrix},
\]

\[
J^{(k,j)} = A^T A + \lambda_L^{(k)} L^T L
\]

\[
\eta_{(k,j)} = b^T b - \lambda_L^{(k)} \delta^2,
\]
Apply Block Gaussian Elimination:

\[
\begin{pmatrix}
J^{(j)} & A^T b \\
0 & \tau_j
\end{pmatrix}
\begin{pmatrix}
x^{(j+1)} \\
-1
\end{pmatrix}
= \beta_j \begin{pmatrix}
x^{(j)} \\
-(z^{(j)})^T x^{(j)} - 1
\end{pmatrix},
\]
Inner Iteration Solve II

Apply Block Gaussian Elimination:

$$
\begin{pmatrix}
  J(j) & A^T b \\
  0 & \tau_j
\end{pmatrix}
\begin{pmatrix}
  x^{(j+1)} \\
  -1
\end{pmatrix}
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  x^{(j)} \\
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\end{pmatrix},
$$

where

$$
\tau_j = \eta_j - b^T Az^{(j)} \quad x^{(j+1)} = z^{(j)} + \beta_j u^{(j)} \quad \beta_j = \tau_j / ((z^{(j)})^T x^{(j)} + 1).
$$
Inner Iteration Solve II

- Apply Block Gaussian Elimination:

\[
\begin{pmatrix}
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-1
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\]

- \(z^{(j)}\) and \(u^{(j)}\) solve \(J(j) z^{(j)} = A^T b\), \(J(j) u^{(j)} = x^{(j)}\).
Practical Details

- **The Sign Condition:** If the condition $g(x^{(k+1)})g(x^{(0)}) > 0$ is relaxed a divergent sequence may result.
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Practical Details

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Practical Details

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- **Do not shift**: For inexact update we do not use the shift because the system matrix $B$ is parameter dependent and it makes no practical sense to force convergence with the shift.
- **Choice of $\gamma$**: The theory is developed for $\gamma = \delta^2$ but the algorithm can use $\gamma = \|Lx^{k,j}\|^2$. If blow up does not occur, the algorithm converges more quickly.
All approaches need to solve normal equations $Jw = f$, 

$$(A^T A + \lambda L^T L)w = f$$
Computational Considerations: Generalized SVD

- All approaches need to solve normal equations $Jw = f$, 
  
  $$ (A^T A + \lambda L L^T)w = f $$

- GSVD For the augmented matrix $[A, L]$, $2m^2 n + 15n^3$ flops.

  $$ A = U \left( \begin{array}{cc} \Sigma & 0 \\ 0 & I_{n-p} \end{array} \right) X^{-1}, \quad L = V \left( \begin{array}{cc} M & 0 \\ 0 & 0 \end{array} \right) X^{-1}. $$

  $U$, $V$, $m \times n$ and $p \times p$, resp., orthonormal. $X$, $n \times n$ nonsingular. $\Sigma$, $M$ diag., $p \times p$, entries $\sigma_i$, $\mu_i$, resp. Then

  $$ \left[ \left( \begin{array}{cc} \Sigma^2 & 0 \\ 0 & I_{n-p} \end{array} \right) + \lambda L \left( \begin{array}{cc} M^2 & 0 \\ 0 & 0 \end{array} \right) \right] X^{-1}w = X^T f. $$
Computational Considerations: Generalized SVD

- All approaches need to solve **normal equations** $Jw = f$,
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- **Efficient** Direct solution can be found efficiently $8n^2$ flops. Solve triangular systems for each $\lambda_L$. 
Inverse Problems

Total Least Squares Methods

Results

Experiments

Conclusions

Other Research
Experiments: Hansen’s Regularization Toolbox

- *ilaplace*, *phillips* and *shaw* - all discretizations of continuous ill-posed Fredholm integral type constructed by quadrature.
- Generate $A$ and $x^*$, calculate $b = Ax^*$ exactly.
- Scale $\|A\|_F = \|b\|_2 = 1$.
- Add noise 5% Gaussian to $b$ and $A$.
- $L$ approxs first derivative, $(n - 1) \times n$.
- Tolerance $\tau = 10^{-4}$.
- Estimated solution $x_{est}$. Relative error with respect to $x^*$.
- Number of system solves $K$. 
Inexact and Exact Algorithms: convergence for $-\lambda_i^{(k)}$

The dotted and dashed lines show convergence for exact and inexact algorithms, resp.. The first row shows the whole convergence history while the second row shows the first 10 steps. Left to right *ilaplace, shaw* and *phillips* resp..
Inclusion of Shift: convergence history of $-\lambda_i$

Using $\gamma = \delta^2$ and $\gamma = \|Lx^{k,j}\|^2$. top and bottom, resp.
shifted, no shift, or shift after first step.

*ilaplace*, *shaw* and *phillips* resp..
Comparison for $\gamma$: $\delta^2$ or $\|Lx^{(k,j)}\|^2$

On top *ilaplace*, and *phillips*. Below *shaw*. Solutions are indicated on the left, and the L-curve on the right.
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Other Research
Numerical experiments have been presented which verify the theory:

- The algorithm provides an efficient and practical approach for the solution of the RTLS problem in which a good estimate of the physical parameter is provided.
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- If no constraint information is provided, the **L-curve** technique can be successfully employed.
Regularized Total Least Squares

Numerical experiments have been presented which verify the theory

- The algorithm provides an **efficient** and practical approach for the solution of the RTLS problem in which a good estimate of the physical parameter is provided.
- If blow up occurs **bisection** search may yield a better solution satisfying the constraint condition.
- If no constraint information is provided, the **L-curve** technique can be successfully employed.
- Algorithm performs better than QEP for all of our tests.
Related Work

- **Domain decomposition (PVDTLS)** for large scale TLS: (2005).
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- **Applications**:
  - **Signal and image restoration** (Geophysical Journal 2006)
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- **Current Directions**
  - Domain Decomposition for large scale RTLS
  - Total Variation TLS
  - Large scale implementation
  - Multiple Right Hand Sides.
Removal of Noise from Difference Images for Longitudinal Studies

▶ **Application:** Two PET scans of the same patient at different times

▶ **Question:** Are there any anatomical or functional changes?
Difference Image after Alignment by Maximization of Mutual Information

- Scans from different days have to be aligned
- Noise and artifacts change from scan to scan.
- Small changes are hard to locate in the difference image
Decomposed Difference Image: wavelets
Difference Image and \( u \) Part

Difference Image

\( u \) Part

Rosemary Renaut with Dr Hongbin Guo and Wolfgang Stefan
u and v Part

u Part

v Part
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Application to Medical Images

Image Registration
THANKYOU

Any Questions