Research Note: A unifying framework for widely-used stabilization of potential field inverse problems

Saeed Vatankhah1,2, Rosemary Anne Renaut3, and Shuang Liu1

1Hubei Subsurface Multi-scale Imaging Key Laboratory, Institute of Geophysics and Geomatics, China University of Geosciences, Wuhan, China.

2Institute of Geophysics, University of Tehran, Iran.

3School of Mathematical and Statistical Sciences, Arizona State University, Tempe, AZ, USA.

(Corresponding author: Shuang Liu, email: lius@cug.edu.cn)

ABSTRACT

We present a brief review of the widely-used and well-known stabilizers in the inversion of potential field data. These include stabilizers that use $L_2$, $L_1$, and $L_0$ norms of the model parameters, and the gradients of the model parameters. These stabilizers may all be realized in a common setting using two general forms with different weighting functions. Moreover, we show that this unifying framework encompasses the use of additional stabilizations which are not common for potential field inversion.

INTRODUCTION

Many forms of stabilizers are used for the inversion of the ill-posed potential field data problem. The aim is the reconstruction of subsurface models that provide relevant physical interpretation. In order to reduce the possibility of over-interpretation of the data, it may be desirable to reconstruct a simple model with as little structure as possible, and to eliminate arbitrary discontinuities in the solution (Constable et al., 1987). Such models can be expected to present only the important and large-scale features of the subsurface under the survey area. They can be obtained using a minimum roughness, or equivalently maximum smoothness, stabilizer that employs a $L_2$-norm of the gradient of the model parameters in the inversion algorithm (Constable et al., 1987; Li and Oldenburg, 1996; Pilkington, 1997; Li and Oldenburg, 1998). A
subsurface model exhibiting discontinuities may also be appropriate and physically relevant (Farquharson and Oldenburg, 1998). These models are achieved by applying a $L_1$-norm total variation (TV) regularization, or an $L_0$-norm minimum gradient support (MGS) stabilizer for the gradient of the model parameters (Portniaguine and Zhdanov, 1999; Bertete-Aguirre et al., 2002; Vatankhah et al., 2018a). The $L_0$-norm, which counts the number of non-zero entries in the vector, does not meet the mathematical requirement to be regarded as a norm but does yield sparsity in the vector. Applying the $L_p$-norm, for $p = 0$ or 1, to the gradient of the model parameters minimizes the number of discontinuous transitions of the reconstructed model. Alternatively, when applied directly to the model parameters, these $L_p$-norms, with $p = 0$ or 1, provide sparsity in the solution, and are relevant when it can be assumed that the sources of interest are localized and compact (Last and Kubik, 1983; Guillen and Menichetti, 1984; Barbosa and Silva, 1994; Portniaguine and Zhdanov, 1999; Zhdanov and Tolstaya, 2004; Ajo-Franklin et al., 2007; Vatankhah et al., 2014a; 2014b; 2015; 2017). Inversion algorithms that employ such sparsity constraints yield sparse and focused images of the subsurface structures. Within the potential field literature, the use of the $L_0$-norm applied for the model parameters is generally referred to as a compactness, or minimum support, constraint (Last and Kubik, 1983; Portniaguine and Zhdanov). Previous investigations have demonstrated that the $L_0$-norm stabilizer can yield a sparser solution than that obtained using the convex $L_1$-norm stabilizer (Chartrand, 2007; Fournier, 2015). In addition, inversion algorithms based on mixed $L_p$-norm stabilizers have been used (Sun and Li, 2014; Fournier and Oldenburg, 2019). These allow simultaneous recovery of both blocky and smooth features of the subsurface targets.

Here we provide a comprehensive overview of the different stabilizers which have been used in the inversion of potential field data. A unifying formulation demonstrates their similarities and minor differences, and provides a common implementation strategy. Hence, this review yields improved insights for selection of appropriate context-specific stabilizers.

**MODEL FORMULATION**
We first review the standard setting for the solution of the potential field linear inversion problem. Suppose \( \mathbf{d}_{\text{obs}} \in \mathbb{R}^N \) is a set of measurements of the potential field, \( \mathbf{m} \in \mathbb{R}^M \) is the vector of unknown model parameters, and \( \mathbf{G} \in \mathbb{R}^{N \times M} \) is the discretization of the forward model operator that connects the data and parameters via the linear set of equations
\[
\mathbf{d}_{\text{obs}} = \mathbf{Gm}.
\] (1)

Note that in general the number of measurements is less than the dimension of the unknown set of parameters so that \( N \ll M \). The aim is to determine a geologically plausible model \( \mathbf{m} \) that predicts \( \mathbf{d}_{\text{obs}} \) at the noise level. Due to the ambiguity in the measurements, and the inherent non-uniqueness of the potential sources, based on Gauss’s theorem, the problem is ill-posed. Hence, solution techniques should incorporate as much information about the formulation and plausible model as possible.

To obtain a physically acceptable solution of equation (1), a global objective function, \( \mathbf{P}^\alpha(\mathbf{m}) \), is constructed. It consists of a linear combination of a data misfit function, \( \Phi(\mathbf{d}) \), and a stabilizing term, \( \Phi(\mathbf{m}) \), which is weighted by a scalar parameter \( \alpha^2 \). Specifically, the objective function to be minimized, subject to the additional physical bound constraints, \( \mathbf{m}_{\text{min}} \leq \mathbf{m} \leq \mathbf{m}_{\text{max}}, \) is given by
\[
\mathbf{P}^\alpha(\mathbf{m}) = \Phi(\mathbf{d}) + \alpha^2 \Phi(\mathbf{m}) = \|W_d(\mathbf{Gm} - \mathbf{d}_{\text{obs}})\|^2_2 + \alpha^2 \|W_{\text{depth}}W_hW_l \mathbf{D}(\mathbf{m} - \mathbf{m}_{\text{appr}})\|^2_2. \] (2)

Equivalently, for given \( \mathbf{D}, \mathbf{m}_{\text{appr}}, \mathbf{W}_h, \mathbf{W}_{\text{depth}}, \mathbf{W}_d, \mathbf{W}_l, \) and constraint bounds \( \mathbf{m}_{\text{min}} \) and \( \mathbf{m}_{\text{max}} \) we seek
\[
\mathbf{m}^* = \arg \min_{\mathbf{m}_{\text{appr}} \leq \mathbf{m} \leq \mathbf{m}_{\text{max}}} \left\{ \|W_d(\mathbf{Gm} - \mathbf{d}_{\text{obs}})\|^2_2 + \alpha^2 \|W_{\text{depth}}W_hW_l \mathbf{D}(\mathbf{m} - \mathbf{m}_{\text{appr}})\|^2_2 \right\}. \] (3)

Here, we note that for a vector \( \mathbf{x} \in \mathbb{R}^n \), we use the standard definition, \( \|\mathbf{x}\|_p = \left(\sum_{i=1}^n |x_i|^p\right)^{1/p}, \ p \geq 1 \), and where \( \|\mathbf{x}\|_0 \) counts the number of nonzero entries in \( \mathbf{x} \). We note that \( \mathbf{m}^* \) is the minimum of \( \mathbf{P}^\alpha(\mathbf{m}) \) subject to the bound constraints \( \mathbf{m}_{\text{min}} \leq \mathbf{m} \leq \mathbf{m}_{\text{max}} \). These constraints can be determined from the available geological information in the survey area and, they have a valuable effect on reducing the non-uniqueness of the solution of the inverse problem. Generally, there are several strategies to impose bound constraints
into the objective function (2). These include, for example, (i) adding a log-barrier penalty term (Li and Oldenburg, 2003), (ii) parametrizing into a quantity that can only be positive (Lelièvre, 2003), and (iii) projection of model parameters outside the bounds to their nearest specified bound (Last and Kubik, 1983; Portniaguine and Zhdanov, 1999; Boulanger and Chouteau, 2001; Vogel, 2002; Fournier, 2015; Vatankhah et al., 2014a; 2015; 2017). All of these strategies make the problem non-linear with respect to \( \mathbf{m} \) and an iterative process is needed to yield the final solution. The third strategy is the simplest and does not require adjustment of additional parameters. Then, the projection to within acceptable bounds is flexible and easily adopted in the minimization of \( \mathbf{P}^\alpha (\mathbf{m}) \).

In equation (2), the misfit function \( \Phi(\mathbf{d}) = \| W_d (\mathbf{Gm} - \mathbf{d}_{\text{obs}}) \|_2^2 \) measures the residual between the observed and predicted data normalized by the weighting matrix \( W_d \in \mathbb{R}^{N \times N} \). Under the assumption that the noise in the data is Gaussian and uncorrelated, \( W_d \) is a diagonal matrix with entries that are estimates of the inverses of the standard deviations of the measurements and serves to whiten the noise in the data. The regularization parameter, \( \alpha \), in equation (2) weights the relative contributions of the two terms \( \Phi(\mathbf{d}) \) and \( \Phi(\mathbf{m}) \). Widely-used automatic algorithms for finding \( \alpha \) are extensively detailed in the literature. These include Generalized Cross Validation (Marquardt, 1970; Golub et al., 1979), the L-curve (Hansen, 1992; Hansen, 2007), unbiased predictive risk estimation (Vogel, 2002), and the Morozov (Morozov, 1966) and \( \chi^2 \)–discrepancy principles (Mead and Renaut, 2009; Vatankhah et al, 2014b). The application of these parameter-choice strategies in focusing inversion algorithms is discussed in Vatankhah et al. (2014a; 2014b; 2015).

The stabilizer controls the growth of the solution with respect to the weighted norm. Here, vector \( \mathbf{m}_{\text{apr}} \in \mathbb{R}^M \) presents an initial estimate of the model, possibly known from a previous investigation. It is also possible to set \( \mathbf{m}_{\text{apr}} = \mathbf{0} \). Generally, the hard constraint matrix \( W_h \in \mathbb{R}^{M \times M} \) is an identity matrix. If, however, additional information is available to set some of the model parameters for specific locations, then this is encoded using \( W_h \). For example, it is possible that prior investigations of the survey area, such
as drilling, or other geological and geophysical information, provide the values of the model parameter for some prisms. Then, this information should be included in \( \mathbf{m}_{\text{data}} \), and this can be achieved by setting the corresponding diagonal entries of \( W_h \) to a large value, for example 100 (Boulanger and Chouteau, 2001; Vatankhah et al., 2014a). The diagonal depth-weighting matrix \( W_{\text{depth}} \in \mathbb{R}^{M \times M} \) is used to counteract the rapid decay of kernels with depth, so that all prisms participate with an approximately equal probability in the inversion algorithm. This limits the tendency for the solution to concentrate near the surface (Li and Oldenburg, 1996; Pilkington, 1997; Boulanger and Chouteau, 2001). For the mean depth \( z_j \) of prism \( j \) and problem-dependent parameter \( \beta \), the \( j \)th entry of the depth-weighting matrix can be written as 

\[
(W_{\text{depth}})_{jj} = \frac{1}{(z_j + z_0)\beta},
\]

where \( z_0 \) depends both upon the prism size and the observation height of the data (Li and Oldenburg, 1996). The parameter \( \beta \) is usually selected based on the data type and dimension of the problem. Note that there are alternative strategies, not considered here, for imposing depth weighting into the inversion algorithm, see for example Zhdanov (2002) and Cella and Fedi (2012).

Without loss of generality, we assume that the inversion takes place in three dimensions. Then, the matrix \( \mathbf{D} = [\alpha_x \mathbf{D}_x; \alpha_y \mathbf{D}_y; \alpha_z \mathbf{D}_z] \in \mathbb{R}^{3M \times M} \) includes discrete approximations for derivatives operators in \( x \), \( y \) and \( z \)-directions, \( \mathbf{D}_x \), \( \mathbf{D}_y \), and \( \mathbf{D}_z \), respectively. It can be assumed that these matrices are square and of the same size with dimension \( M \). For example, if discrete derivatives are defined at the centers of each prism, the first order difference operators will have the same size (Li and Oldenburg, 2000). Further, if we use forward differences for all prisms and backward differences where necessary, it is possible to make the operators for the first order derivatives square with dimension \( M \) (Lelièvre and Oldenburg, 2009; Lelièvre and Farquharson, 2013). Moreover, for higher order derivatives, square matrices are also achievable by suitable combinations of central, forward or backward differencing away from the boundaries of the volume, combined with suitable one-sided approximations at the boundaries. Note also that these derivative approximations can be of different orders, for example a zeroth order approximation may be applied in the
z-dimension by taking \( D_z = I \), while taking \( D_x \) and \( D_y \) as approximations to the first order derivatives. The adjustable constants \( \alpha_x, \alpha_y \) and \( \alpha_z \) control the relative contributions between the various components (Li and Oldenburg, 1996; 1998).

The choice of \( W_L \) in stabilizer \( \Phi(m) \) is the focus of our discussion. It is \( \Phi(m) \) which controls the growth of the solution \( m^+ \) with respect to the weighted norm, and, hence, determines the characteristics of the estimated subsurface structure. Stabilization with respect to different \( L_p \)-norm stabilizers, for arbitrary \( p \geq 0 \), is achieved using appropriate choices of the weighting matrix \( W_L \), in \( \Phi(m) \), such that the resulting \( L_2 \)-norm in \( \Phi(m) \) generates an approximation to an \( L_p \)-norm (Wohlberg and Rodriguez, 2007). It is the introduction of \( W_L \), along with using the matrix \( D \), that provides a common framework for the different stabilizers that have been used in potential field inversion.

When \( D \) is replaced by the identity matrix, \( D = I \), and the weighting matrix \( W_L \) is selected as

\[
W_L = \text{diag}\left( \frac{1}{((m - m_{\text{apr}})^2 + \varepsilon^2)^{\frac{2-p}{4}}} \right) \in \mathbb{R}^{M \times M},
\]

we have \( \Phi(m) = \| W_{\text{depth}} W_h W_L (m - m_{\text{apr}}) \|^2_2 \). Now, the choice \( p = 2 \) provides a \( L_2 \)-norm solution of the model parameters, while \( p = 0 \) and \( p = 1 \) provide the compact \( L_0 \)-norm and \( L_1 \)-norm solutions, respectively. Note that with the \( L_2 \) measure, \( p = 2 \), the solution of the inverse problem exhibits some smoothness, although less than that with the minimum-structure inversion. When it can be assumed that the sources of interest are localized and compact, it is more appropriate to take \( p = 1 \) or \( p = 0 \) (Last and Kubik, 1983; Portniaguine and Zhdanov, 1999; Vatankhah et al., 2017). In the potential field literature, the choice with \( p = 0 \) is referred to as the solution which is compact and has minimum support; the number of nonzero entries in the solution is minimized and \( m \) is constrained to be sparse. In equation (4), the focusing parameter \( 0 < \varepsilon << 1 \) is added to avoid the possibility of division by zero and assures that \( W_L \) is invertible. The choice of \( \varepsilon \) has an important effect on the solution. Small values of \( \varepsilon \) lead to models with high
contrast, but also increase the instability in the solution. On the other hand, for large value of $\varepsilon$ the image will not be focused (Last and Kubik, 1983; Portniaguine and Zhdanov, 1999; Farquharson and Oldenburg, 1998; Ajo-Franklin et al., 2007; Vatankhah et al., 2017). In general, we are interested in the case where $\varepsilon \to 0$, because in that case the stabilizer counts the number of vector elements different from zero. In many research papers, it has been suggested that a pre-defined small fixed $\varepsilon$ can be used (Last and Kubik, 1983; Ajo-Franklin et al., 2007; Vatankhah et al., 2017). On the other hand, Zhdanov and Tolstaya (2004) suggested an approach for automatic estimation of $\varepsilon$ as the point of maximum curvature of the plot of the model objective function as a function of $\varepsilon$. Alternatively, Fournier (2015) and Fournier and Oldenburg (2019) applied a $\varepsilon$-cooling strategy, in which the focusing parameter is initialized as a large value, and then monotonically reduced during the iterations until the inversion algorithm satisfies the convergence criteria. Further, they suggested that $\varepsilon$ can be decreased until it reaches a pre-defined target value. For more discussion on the focusing parameter $\varepsilon$, and the strategies which have been proposed for selecting this parameter, see Fiandaca et al. (2015). We also note that the solutions using $L_1$, rather than $L_0$, stabilization, are less sensitive to $\varepsilon$, due to the use of the fourth root ($p = 1$), instead of the square root ($p = 0$), in the calculation of $W_L$.

We now discuss the choices for the stabilization function $\Phi(m)$ when $D = [\alpha_x D_x; \alpha_y D_y; \alpha_z D_z]$. Then

$$W_L = \begin{bmatrix} W_R & 0 & 0 \\ 0 & W_R & 0 \\ 0 & 0 & W_R \end{bmatrix} \in \mathbb{R}^{3M \times 3M} \text{ is block diagonal with}$$

$$W_R = \text{diag} \left( \frac{1}{((\alpha_x D_x (m - m_{\text{apr}}))^2 + (\alpha_y D_y (m - m_{\text{apr}}))^2 + (\alpha_z D_z (m - m_{\text{apr}}))^2 + \varepsilon^2)^{\frac{2-p}{4}}} \right) \in \mathbb{R}^{M \times M} . \quad (5)$$

First, we note that, to make the multiplication in equation (2) dimensionally consistent, the matrix $W_{\text{depth}} W_h \in \mathbb{R}^{M \times M}$ is replaced with

$$W_{\text{depth}} W_h \in \mathbb{R}^{M \times M} \text{ is replaced with}$$

$$\begin{bmatrix} W_{\text{depth}} W_h & 0 & 0 \\ 0 & W_{\text{depth}} W_h & 0 \\ 0 & 0 & W_{\text{depth}} W_h \end{bmatrix} \in \mathbb{R}^{3M \times 3M} .$$

Now, for $p = 2$, the
application of (5) in $\Phi(m)$ generates the solution with minimum structure and the inversion methodology produces a smooth image of the subsurface. This eliminates arbitrary discontinuities in the solution and should reduce any temptation to over interpret the data. But, if we anticipate that there are true discontinuous jumps in the subsurface model, a smooth model will not be satisfactory. It may, then, be appropriate to take $p = 1$ or $p = 0$, for a total variation (TV) or minimum gradient support (MGS) regularization, respectively (Portniaguine and Zhdanov, 1999; Bertete-Aguirre et al., 2002; Farquharson and Oldenburg, 1998; Vatankhah et al., 2018a). In both cases, the regularization term keeps $Dm$ sparse, so that the number of discontinuous transitions is kept to a minimum.

It is immediate that while the noted choices with $p = 0, 1, 2$ provide stabilizers that are widely used in the potential field arena, the formulation is general. For example, selecting $p = 3/2$ in (4) yields a stabilizer that minimizes the model parameters measured for the $L_{3/2}$-norm. Further, the formulation explains the role of the weighting matrix $W_L$ in converting the non-quadratic $L_p$-norm stabilizer to a pseudo-quadratic function. For stabilizers based on $p = 2$, i.e. the $L_2$-norm of model parameters, or of the gradient of the model parameters, $W_L = I$, and (2) is quadratic and differentiable. On the other hand, for $p = 0$, or 1, the original form of the $L_p$-norm stabilizer, $\Phi(m) = \|W_{depth} W_n D(m - m_{apr})\|_p$, is not differentiable. But, recasting the $L_p$-norm stabilizer using the $L_2$-norm approximation with the appropriate choice of $W_L$ in $\Phi(m) = \|W_{depth} W_n W_L D(m - m_{apr})\|^2_{L_2}$ yields a differentiable quadratic function. This is a widely-used and well-known strategy for handling the non-differentiability when $p < 2$. In fact, the resulting stabilizers are now pseudo-quadratic in $m$ and differentiable. Dependent on $p$ and $D$, $W_L$ is modified to achieve the appropriate approximation (Wohlberg and Rodriguez, 2007).

**Numerical strategies for the solution**
The minimization of the objective function introduced by equation (2), for a given iterate, can be treated as the minimization of the conventional Tikhonov functional. Setting \( W = W_{\text{depth}}W_hW_L \), for \( W_L \) at the current iterate, in equation (2), in which the dimension of \( W \) is \( M \times M \) when \( D = I \) and \( 3M \times 3M \) for \( D \neq I \), and differentiating with respect to \( m \), then the iterative update for the given \( \alpha \) is

\[
\mathbf{m}(\alpha) = \mathbf{m}_{\text{apr}} + (G^TW_d^TW_dG + \alpha^2D^TW^TW)\mathbf{d}_G(\mathbf{d}_\text{obs} - G\mathbf{m}_{\text{apr}}).
\]  

(6)

Here it is assumed that the null spaces of \( WD \) and \( W_dG \) do not intersect. Now, however, because the minimum needs to be obtained with respect to bound constraints on \( m \), and because equation (2) is nonlinear in \( m \) through the definition of \( W_L \), an iterative approach needs to be applied to find \( \mathbf{m}^* \) as the solution of equation (2). An iteratively reweighted least square (IRLS) approach is used. At iteration \( k \), \( W_L \) is estimated using the model parameters obtained at the previous iteration (Farquharson and Oldenburg, 1998; Portniaguine and Zhdanov, 1999; Wohlberg and Rodriguez 2007; Vatankhah et al., 2017, 2018b). Computationally, we note that both (4) and (5) are by definition invertible, and obtained at zero cost. Thus, the solution at each iteration \( k \) requires only (i) the update for the entries in \( W_L \), (ii) the estimation of a suitable regularization parameter \( \alpha \), and (iii) the solution (6) of a general Tikhonov regularized problem. Therefore, for small scale problems, it is feasible to use the singular value decomposition (SVD) (when \( D = I \)) or the generalized singular value decomposition (GSVD), (\( D \neq I \)), Hansen (2007). For large-scale problems with \( D = I \), the randomized SVD or Golub-Kahan Bidiagonalization (GKB) can be used (Vatankhah et al., 2017, 2018b). If the problem is large scale and \( D \neq I \), the randomized GSVD, or joint GKB can be used (Vatankhah et al., 2018a). Moreover, it is also possible to solve using an alternating direction minimization, to avoid requiring a randomized GSVD for matrix \( W \) of size \( 3M \times 3M \) at each iteration. Instead, by iterating over dimensions \( x, y \) and \( z \), independently for each update \( k \), the computation requires three approximate GSVD calculations for matrices of size \( M \times M \) (Vatankhah et al., 2018a). Thus, this unifying framework provides an effective
and computationally straightforward approach for general stabilization in the inversion of potential field data.

**CONCLUSION**

A unifying, yet simple, framework of the widely-used stabilizers in the inversion of potential field data has been presented. These stabilizers differ only through the definition of the weighting that is applied. Moreover, the unifying framework encompasses the use of additional stabilizations which are not common for potential field inversion.

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