DEPTH FORMULAS, RESTRICTED TOR-DIMENSION UNDER BASE CHANGE

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ABSTRACT. Let $R$ be a commutative Noetherian ring, and let $M$ and $N$ be $R$-modules. It is shown that
\[
\sup\{i \mid \text{Tor}_i^R(M, N) \neq 0\} = \sup\{\text{depth}_p R - \text{depth}_p M_p - \text{depth}_p N_p \mid p \in \text{Supp } M \cap \text{Supp } N\}
\]
provided that $M$ has finite flat dimension. Assume that $R$ is a complete local ring, $M$ a finitely generated $R$-module, and $N$ an $R$-module of finite flat dimension. It is then proved that
\[
\sup\{i \mid \text{Ext}_i^R(N, M) \neq 0\} = \text{depth } R - \text{depth } N.
\]
Set
\[
\text{Td}_R M = \sup\{i \in \mathbb{N}_0 \mid \text{Tor}_i^R(T, M) \neq 0 \text{ for some } T \text{ of finite flat dimension}\}.
\]
In addition, some results concerning $\text{Td}_R M$ under base change are given.

1. Introduction. Throughout this paper all rings are assumed to be commutative and Noetherian. It is well known that flat dimension of an $R$-module $M$ can be computed by the following formula
\[
\text{fd}_R M = \sup\{i \in \mathbb{N}_0 \mid \text{Tor}_i^R(T, M) \neq 0 \text{ for some } R\text{-module } T\}.
\]
If flat dimension of $M$ is finite, then it can be computed by Chouinard’s formula [5, Corollary 1.2]:
\[
\text{fd}_R M = \sup\{\text{depth}_p R - \text{depth}_p M_p \mid p \in \text{Spec } R\}.
\]
Foxby has studied the restricted Tor-dimension: $\text{Td}_R M = \sup\{i \in \mathbb{N}_0 \mid \text{Tor}_i^R(T, M) \neq 0 \text{ for some } T \text{ with } \text{fd}_R T < \infty\}$. Over a ring of finite Krull dimension it is easy to see that $\text{Td}_R M \leq \dim R < \infty$ for any $R$-module $M$. In this case $\text{Td}_R M$