MONADS AND BUNDLES ON RATIONAL SURFACES

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ABSTRACT. A monad construction is presented for holomorphic bundles on an arbitrary blowup of \( \mathbb{P}_2 \) which have semi-stable direct image on \( \mathbb{P}_2 \). Three illustrative applications to different moduli problems are given.

1. Introduction. A monad \( M \) on a complex manifold \( X \) is a complex \( 0 \rightarrow A \rightarrow^a B \rightarrow^b C \rightarrow 0 \) of holomorphic vector bundles with \( a(x) \) injective and \( b(x) \) surjective at each \( x \in X \); the cohomology of \( M \) is the vector bundle \( E(M) = \text{Ker} b / \text{Im} a \). The utility of monads lies in the fact that, under certain auspicious conditions, a vector bundle (or family of such) can be described as the cohomology of a monad (or family of such) of a particularly simple kind.

Horrocks [13] was the first to introduce monads and used them to show that every holomorphic vector bundle on \( \mathbb{P}_n \) can be described by monads with \( A, B, C \) all projectively trivial, i.e., trivial twisted by a line bundle. Barth [3] used this to classify stable bundles on \( \mathbb{P}_2 \) up to linear algebraic data, and this work was extensively generalized and developed in the book [15]. The monad description of bundles on \( \mathbb{P}_3 \) was used by Atiyah et al. [2] in their celebrated description of instantons (self-dual solutions of the Yang-Mills equations) on \( S^4 \), using the Ward correspondence [17] between holomorphic bundles on \( \mathbb{P}_3 \) and the instantons on \( S^4 \). The close relationship between complex analytic geometry and gauge theory has provided a rich source for applications of monads, particularly in the context of computing moduli spaces.

Methods similar to those used for the ADHM construction were used in [7] to describe the instantons on \( \mathbb{C}\mathbb{P}_2 \); in this case, instantons correspond to certain holomorphic vector bundles on the flag manifold.