Slopes and Curvature  

Consider a curve, such as \( f(x) = x^2 \). The instantaneous rate of change at a point on the curve is the slope of the line tangent to the curve at that point. Shown below is the graph of \( y = x^2 \) with six tangent line segments at \( x_1 = -2, x_2 = -1, x_3 = -0.5, x_4 = 0.25, x_5 = 1, \) and \( x_6 = 2 \). Since the derivative is \( f'(x) = 2x \), the six corresponding slopes are

\[
\begin{align*}
    m_1 &= f'(-2) = -4; \\
    m_2 &= f'(-1) = -2; \\
    m_3 &= f'(-0.5) = -1; \\
    m_4 &= f'(0.25) = 0.5; \\
    m_5 &= f'(1) = 2; \\
    m_6 &= f'(2) = 4; \\
\end{align*}
\]

Now since \( m = \frac{\Delta y}{\Delta x} \) is a constant on a line, and \( \tan(\theta) = \frac{\text{opp}}{\text{adj}} \) in any right triangle, we have that \( \tan(\theta) = \frac{\Delta y}{\Delta x} \) or equivalently, that \( \tan(\theta) = m \) gives a relationship between the angle of elevation a line makes with the horizontal and the slope of that line.

That is, \( \theta = \tan^{-1} \left( \frac{\Delta y}{\Delta x} \right) = \tan^{-1} (m) \). And since we are considering tangent lines, this means the slope came from the derivative. In particular, for the curve above,

\[
\begin{align*}
    \theta_1 &= \tan^{-1} (-4) \approx -1.33; \\
    \theta_2 &= \tan^{-1} (-2) \approx -1.11; \\
    \theta_3 &= \tan^{-1} (-1) = -\frac{\pi}{4}; \\
    \theta_4 &= \tan^{-1} (0.5) \approx 0.464; \\
    \theta_5 &= \tan^{-1} (2) \approx 1.11; \\
    \theta_6 &= \tan^{-1} (4) \approx 1.33; \\
\end{align*}
\]

Notice that the angle is changing much quicker as we move along the curve near the vertex of the parabola than when it is farther from the vertex. So we can intuitively see that the
curvature is greater at the vertex, and less so the more we move away from the vertex. In fact as the parabola gets steeper, the tangent line angle monotonically approaches $\frac{\pi}{2} \approx 1.57$.

We should also notice that the tangent lines are a better fit the lower the curvature. The tangent line is the line of best fit, but for $f(x) = x^2$, how well it ‘fits’ at the origin (the x-axis is the tangent line) is not as good as how well it fits at $x = \pm 2$.

**The Math Details**  We want to find $\frac{d\theta}{dx}$. By the chain rule, $\frac{d\theta}{dx} = \frac{d\theta}{ds} \frac{ds}{dx}$. Now since at any $x$-value we have $\theta = \tan^{-1} (f'(x))$, this gives us

$$\frac{d\theta}{dx} = \frac{f''(x)}{1 + f'(x)^2}$$

Also, the arc length formula is

$$s = \int_a^b \sqrt{1 + f'(x)^2} \, dx \Rightarrow ds = \sqrt{1 + f'(x)^2} \, dx \Rightarrow \frac{dx}{ds} = \frac{1}{\sqrt{1 + f'(x)^2}}$$

Therefore

$$\frac{d\theta}{ds} = \frac{d\theta}{dx} \frac{dx}{ds} = \frac{f''(x)}{1 + f'(x)^2} \frac{1}{\sqrt{1 + f'(x)^2}} = \frac{f''(x)}{(1 + f'(x)^2)^{3/2}}$$

We define

$$\kappa(x) = |\frac{d\theta}{ds}| = \frac{|f''(x)|}{(1 + f'(x)^2)^{3/2}}$$

(Without the absolute value sign, the curvature could be negative, but this just means the circle of best fit has its center below the curve, instead of above it.)

As an exercise, it is not hard to show that for

$$f(x) = \pm \sqrt{a^2 - x^2}, \quad \kappa(x) = \left| \frac{\sqrt{a^2 - x^2}}{(a^2 - x^2)^{3/2}} \right| = \frac{1}{a}$$

That is, a circle has constant curvature! And the reciprocal of the curvature is the radius of the circle! In fact, for any curve, at each point there is a circle of best fit with radius $a = \frac{1}{\kappa}$, and center in a direction orthogonal to the tangent line.

Incidently, the maximum curvature of polynomials with more than one local extrema is not quite where that local extrema is, although it will be very close. For example, the curve $f(x) = x^3 - 3x$ has its local maximum and local minimum at $x = -1$ and $x = 1$, respectively. The curvature of $f$ is (omitting the absolute value temporarily)

$$\kappa(x) = \frac{6x}{(1 + (3x^2 - 3)^2)^{3/2}} \Rightarrow \kappa'(x) = \frac{-6(45x^4 - 36x^2 - 10)}{(10x^4 - 18x^2)^{5/2}}$$

Solving $\kappa'(x) = 0$ leads to the two solutions $\pm \sqrt{\frac{90 + 15\sqrt{86}}{15}} \approx \pm 1.009079399$, and the curvature at these points will be $|\pm 6.027380992|$. Note the critical points of $k(x)$ are very close to the critical points of $f$ which are $\pm 1$, with the curvature there being $\kappa(\pm 1) = |\pm 6|$. 