Homework 2 Solutions

Chapter 2 #11:

(a) If \( u \) is a scalar operator, then it has a closed invariant subspace different from 0, \( H \) because \( \dim H > 1 \). On the other hand, if \( u \) is not scalar, then \( \sigma(u) \) has two distinct points \( \lambda_1, \lambda_2 \) (since the Gelfand transform of \( u \) is not a constant function). Let \( U_1 \) and \( U_2 \) be disjoint relatively open subsets of \( \sigma(u) \) containing \( \lambda_1, \lambda_2 \) (respectively).

Claim: \( E(U_i) \neq 0 \) for \( i = 1, 2 \). Without loss of generality it suffices to show it for \( i = 1 \): by Urysohn’s Lemma there exists a nonzero \( f \in C(\sigma(u)) \) such that \( 0 \leq f \leq U_1 \). Since the continuous functional calculus is injective, we have \( f(u) \neq 0 \).

Since \( 0 \leq f(u) \leq f(\chi_{U_1}) = E(U_1) \), we have \( E(U_1) \neq 0 \).

Thus \( E(U_1), E(U_2) \) are nonzero mutually orthogonal projections which commute with \( u \), hence their ranges are nonzero invariant subspaces. In particular, \( \text{ran } E(U_1) \) is a closed invariant subspace different from 0, \( H \).

(b) Since \( \lambda \) is isolated in \( \sigma(u) \), the set \( V := \{ \lambda \} \) is nonempty and relatively open in \( \sigma(u) \). Thus, as in part (a), \( \text{ran } E(V) \) is a nontrivial closed invariant subspace. Since \( z\chi_V(z) = \lambda \chi_V(z) \), we have \( uE(V) = \lambda E(V) \), so \( \text{ran } E(V) \subset \ker(u - \lambda) \).

In particular, \( \lambda \) is an eigenvalue of \( u \).

On the other hand, define \( f : \sigma(u) \to \mathbb{C} \) by

\[
f(z) = \begin{cases} 
\frac{1}{z - \lambda} & \text{if } z \neq \lambda \\
0 & \text{if } z = \lambda
\end{cases}
\]

Since \( \lambda \) is isolated in \( \sigma(u) \), we have \( f \in C(\sigma(u)) \). Let \( W = \sigma(u) \setminus V \). Then \( g(z)(z - \lambda) = \chi_W(z) \) for all \( z \in \sigma(u) \), so \( E(W) = g(u)(u - \lambda) \). Thus \( \ker(u - \lambda) \subset \ker E(W) = \text{ran } E(V) \).

Chapter 3 #4: We have

\[
\begin{pmatrix} a & b \\
\alpha(b) & \alpha(a) \end{pmatrix} + \begin{pmatrix} c & d \\
\alpha(d) & \alpha(c) \end{pmatrix} = \begin{pmatrix} a + c & b + d \\
\alpha(b + d) & \alpha(a + c) \end{pmatrix} \in B,
\]

\[
\lambda \begin{pmatrix} a & b \\
\alpha(b) & \alpha(a) \end{pmatrix} = \begin{pmatrix} \lambda a & \lambda b \\
\alpha(\lambda b) & \alpha(\lambda a) \end{pmatrix} \in B,
\]
\[
\left( \begin{array}{cc}
  a & b \\
  \alpha(b) & \alpha(a)
\end{array} \right) \left( \begin{array}{cc}
  c & d \\
  \alpha(d) & \alpha(c)
\end{array} \right) = \left( \begin{array}{cc}
  ac + ba(d) & ad + ba(c) \\
  \alpha(b)c + \alpha(ad) & \alpha(b)d + \alpha(ac)
\end{array} \right) = \left( \begin{array}{cc}
  ac + ba(d) & ab\alpha(c) \\
  \alpha(ad + ba(c)) & \alpha(ac + ba(d))
\end{array} \right) \in B
\]

since \(\alpha^2 = \text{id}\), and
\[
\left( \begin{array}{cc}
  a & b \\
  \alpha(b) & \alpha(a)
\end{array} \right)^* = \left( \begin{array}{cc}
  a^* & \alpha(b)^* \\
  b^* & \alpha(a^*)
\end{array} \right) = \left( \begin{array}{cc}
  a^* & \alpha(b^*) \\
  \alpha(a(b^*)) & \alpha(a(a^*))
\end{array} \right) \in B,
\]
again since \(\alpha^2 = \text{id}\). Thus \(B\) is a \(*\)-subalgebra of \(M_2(A)\).

If \(\left( \begin{array}{cc}
  a_n & b_n \\
  \alpha(b_n) & \alpha(a_n)
\end{array} \right) \to \left( \begin{array}{cc}
  c & d \\
  e & f
\end{array} \right) \) in \(M_2(A)\), then \(a_n \to c\) and \(b_n \to d\), so \(\alpha(a_n) \to \alpha(c)\) and \(\alpha(b_n) \to \alpha(d)\), so
\[
\left( \begin{array}{cc}
  c & d \\
  \alpha(d) & \alpha(c)
\end{array} \right) \in B.\]
Thus \(B\) is closed in \(M_2(A)\), and is therefore a \(C^*\)-subalgebra.

For the map \(\phi\), we have
\[
\phi(a) + \phi(b) = \left( \begin{array}{cc}
  a & 0 \\
  0 & \alpha(a)
\end{array} \right) + \left( \begin{array}{cc}
  b & 0 \\
  0 & \alpha(b)
\end{array} \right) = \left( \begin{array}{cc}
  a + b & 0 \\
  0 & \alpha(a + b)
\end{array} \right) = \phi(a + b),
\]
\[
\lambda\phi(a) = \lambda \left( \begin{array}{cc}
  a & 0 \\
  0 & \alpha(a)
\end{array} \right) = \left( \begin{array}{cc}
  \lambda a & 0 \\
  0 & \alpha(\lambda a)
\end{array} \right) = \phi(\lambda a),
\]
\[
\phi(\phi(a)) = \left( \begin{array}{cc}
  a & 0 \\
  0 & \alpha(a)
\end{array} \right) \left( \begin{array}{cc}
  b & 0 \\
  0 & \alpha(b)
\end{array} \right) = \left( \begin{array}{cc}
  ab & 0 \\
  0 & \alpha(ab)
\end{array} \right) = \phi(ab),
\]
and
\[
\phi(a)^* = \left( \begin{array}{cc}
  a & 0 \\
  0 & \alpha(a)
\end{array} \right)^* = \left( \begin{array}{cc}
  a^* & 0 \\
  0 & \alpha(a^*)
\end{array} \right) = \phi(a^*),
\]
so \(p\) is a homomorphism (recall that we tacitly mean \(*\)-homomorphism when we talk of homomorphisms between \(C^*\)-algebras). If \(\left( \begin{array}{cc}
  a & 0 \\
  0 & \alpha(a)
\end{array} \right) = \left( \begin{array}{cc}
  0 & 0 \\
  0 & 0
\end{array} \right)\), then \(a = 0\), so \(\phi\) is injective.

For the map \(\psi\), define \(\psi' : B \to C\) by \(\psi'(a + bu) = \psi(a) + \psi(b)v\) (which is well-defined since the matrix
\[
\left( \begin{array}{cc}
  a & 0 \\
  0 & \alpha(a)
\end{array} \right) + \left( \begin{array}{cc}
  b & 0 \\
  0 & \alpha(b)
\end{array} \right) \left( \begin{array}{cc}
  0 & 1 \\
  0 & 0
\end{array} \right) = \left( \begin{array}{cc}
  a & b \\
  \alpha(b) & \alpha(a)
\end{array} \right)
\]
uniquely determines \(a\) and \(b\). Then
\[
\psi'(a + bu) + \psi'(c + du) = \psi(a) + \psi(b)v + \psi(c) + \psi(d)v
\]
\[
= \psi(a + b) + \psi(b + d)v
\]
\[
= \psi'(a + c + (b + d)u),
\]
\[
\psi'(\lambda(a + bu)) = \psi'(\lambda a + \lambda bu) \\
= \psi(\lambda a) + \psi(\lambda b)v \\
= \lambda(\psi(a) + \psi(b)v) \\
= \lambda(\psi'(a + bu)),
\]

\[
\psi'(a + bu)\psi'(c + du) \\
= (\psi(a) + \psi(b)v)(\psi(c) + \psi(d)v) \\
= \psi(a)\psi(c) + \psi(a)\psi(d)v + \psi(b)v\psi(c) + \psi(b)v\psi(d)v \\
= \psi(ac) + \psi(ad)v + \psi(b)v\psi(c)v^2 + \psi(b)v(\alpha(d)) \\
= \psi(ac) + \psi(ad)v + \psi(b)\psi(\alpha(c))v + \psi(b\alpha(d)) \\
= \psi(ac) + \psi(ad)v + \psi(b\alpha(c))v + \psi(b\alpha(d)) \\
= \psi'(ac + adu + b\alpha(c)u + b\alpha(d)) \\
= \psi'((a + adu + buc + budu) \\
= \psi'((a + bu)(c + du)),
\]

and

\[
\psi'((a + bu)^*) = \psi'(a^* + ub^*) \\
= \psi'(a^* + \alpha(b^*)u) \\
= \psi(a^*) + \psi(\alpha(b^*))v \\
= \psi(a^*) + v\psi(b)^* \\
= (\psi(a) + v\psi(b))^* \\
= \psi'(a + bu)^*,
\]

so \(\psi'\) is a homomorphism with \(\psi' \circ \phi = \psi\) and \(\psi'(u) = v\).

For the uniqueness, if \(\psi''\) is any such homomorphism, then

\[
\psi''(a + bu) = \psi''(a) + \psi''(b)\psi''(u) \\
= \psi(a) + \psi(b)v \\
= \psi'(a + bu),
\]

so \(\psi'' = \psi'\).