Notes on: $f$ in $A \times _{\alpha } G$

Let $(A, G, \alpha )$ be an action (where $A$ is a $C^*$-algebra and $G$ is a locally compact group), and let $f \in C_c(G, A)$. Also let $(i_A, i_G)$ be the canonical covariant homomorphism of $(A, G)$ in $M(A \times _{\alpha } G)$. We are concerned with the proof of the following:

**Proposition 1.** $f = i_A \times i_G(f)$.

The proof in [1, Corollary 2.36] is not quite adequate because, when it refers to [1, Lemma 2.31], it is to conclude that if $T \in M(A \times _{\alpha } G)$ and $\pi \times U(T) = 0$ for all nondegenerate covariant representations $(\pi, U)$ then $Tf = 0$ for all $f \in A \times _{\alpha } G$. Of course this would suffice to show that $T = 0$, since a multiplier is determined by its action as a left multiplier. However, Lemma 2.31 only implies that if $f \in C_c(G, A)$ and $\pi \times U(f) = 0$ for all nondegenerate covariant representations then $f = 0$ — it does not show the corresponding conclusion for all $f \in A \times _{\alpha } G$. To fix this small gap we need to slightly strengthen the last statement of Lemma 2.31:

**Lemma 2.** If $f \in A \times _{\alpha } G$ then

$$\|f\| = \sup\{\|\pi \times U(f)\| : (\pi, u) \text{ is a nondegenerate covariant representation}\}.$$  

**Proof.** Let $\varepsilon > 0$. It suffices to show that there exists a nondegenerate covariant representation $(\pi, U)$ such that

$$\|\pi \times U(f)\| > \|f\| - \varepsilon.$$ 

Choose $g \in C_c(G, A)$ such that $\|f - g\| < \varepsilon /3$. Then by [1, Lemma 2.31] we can choose a nondegenerate covariant representation $(\pi, U)$ such that

$$\|\pi \times U(g)\| > \|g\| - \frac{\varepsilon}{3}.$$ 

We have

$$\|\pi \times U(f)\| \geq \|\pi \times U(g)\| - \|\pi \times U(f) - \pi \times U(g)\|$$

$$> \|g\| - \frac{\varepsilon}{3} - \|\pi \times U(f - g)\|$$

$$\geq \|f\| - \|g - f\| - \frac{\varepsilon}{3} - \|f - g\|$$

$$> \|f\| - \varepsilon \quad \Box$$

**References**