Tensor Products

This homework assignment is somewhat different from our previous ones — it requires a nontrivial amount of preparation.

Read about tensor products in, for example, the Murphy book. We particularly need tensor products of $C^*$-algebras and of Hilbert spaces. Here I'll summarize the facts you need to know:

In all cases, the tensor product $V \otimes W$ of Banach spaces $V$ and $W$ is a completion of the algebraic tensor product $V \odot W$, where the latter is by definition the linear span of the elementary tensors $v \otimes w$ for $v \in V$ and $w \in W$. You probably saw tensor products of abelian groups in your algebra course. You might have also seen tensor products of modules (over a group or a ring). You might even have seen tensor products of vector spaces. An algebraic tensor product of vector spaces is the same as a tensor product of $C$-modules. The completion is with respect to a tensor norm on the algebraic tensor product, which means we require

$$\|v \otimes w\| = \|v\| \|w\|$$

on elementary tensors.

For Hilbert spaces $H, K$, the algebraic tensor product $H \odot K$ becomes a pre-Hilbert space with the inner product given on elementary tensors by

$$\langle v \otimes w, u \otimes z \rangle = \langle v, u \rangle \langle w, z \rangle,$$

and the Hilbert space tensor product $H \otimes K$ is the completion in the associated norm. For $C^*$-algebras $A, B$, the algebraic tensor product $A \odot B$ becomes a $*$-algebra with multiplication and involution given on elementary tensors by

$$(a \otimes b)(c \otimes d) = ac \otimes bd \quad \text{and} \quad (a \otimes b)^* = a^* \otimes b^*.$$

A $C^*$-tensor product $A \otimes_\alpha B$ is a completion of $A \odot B$ relative to any $C^*$-norm $\| \cdot \|_\alpha$. Interestingly, in general there are many $C^*$-norms on the algebraic tensor product. We need to know about the two extremes:

1. The minimal, or spatial, $C^*$-norm is characterized as follows: if $A$ and $B$ are faithfully and nondegenerately represented on Hilbert spaces $H$ and $K$, then $A \odot B$ becomes a $*$-subalgebra of $B(H \otimes K)$, and the minimal $C^*$-norm is the associated operator norm. The minimal $C^*$-norm deserves its name because it is in fact the smallest $C^*$-norm on $A \odot B$, that is, if $\| \cdot \|_{\text{min}}$ is the minimal norm and $\| \cdot \|_\alpha$ is any $C^*$-norm on $A \odot B$, then for all $d \in A \odot B$ we have $\|d\|_{\text{min}} \leq \|d\|_\alpha$. The completion of

\[ \text{tensor product} V \otimes W \quad \text{algebraic tensor product} V \odot W \quad \text{elementary tensors} v \otimes w \quad \text{tensor norm} \| \cdot \| \]

\[ \text{tensor product} H \odot K \quad \text{Hilbert space tensor product} H \otimes K \quad \text{minimal} \| \cdot \|_{\text{min}} \quad \text{spatial} \| \cdot \|_\alpha \]

\[ (a \otimes b)(c \otimes d) = ac \otimes bd \quad (a \otimes b)^* = a^* \otimes b^* \]

\[ A \otimes_\alpha B \]

\[ \text{minimal} \| \cdot \|_{\text{min}} \leq \|d\|_\alpha \]

\[ \text{and this is a can of worms} \]

\[ \text{where an elementary tensor } T \otimes S \text{ of operators becomes a bounded linear operator on } H \otimes K \text{ in the unique manner determined by } (T \otimes S)(\xi \otimes \eta) = T\xi \otimes S\eta \]
A \otimes B in the minimal norm is called the **minimal tensor product** and sometimes denoted by \( A \otimes_{\text{min}} B \). The minimal tensor product is characterized by the following universal property: if \( \pi : A \to B(H) \) and \( \rho : B \to B(K) \) are representations, then there is a unique representation \( \pi \otimes \rho : A \otimes_{\text{min}} B \to B(H \otimes K) \) such that

\[
(\pi \otimes \rho)(a \otimes b) = \pi(a)\rho(b) \quad \text{for all } a \in A, b \in B.
\]

(2) The **maximal** \( C^* \)-norm is defined by

\[
\|d\|_{\text{max}} := \sup\{\|d\| : \| \cdot \| \text{ is a } C^*\text{-norm on } A \otimes B\}.
\]

The completion of \( A \otimes B \) in the maximal norm is called the **maximal tensor product** and is sometimes denoted by \( A \otimes_{\text{max}} B \). The maximal tensor product is characterized by the following universal property: if \( C \) is any \( C^* \)-algebra and \( \pi : A \to C \) and \( \rho : B \to C \) are homomorphisms which **commute** in the sense that

\[
\pi(a)\rho(b) = \rho(b)\pi(a) \quad \text{for all } a \in A, b \in B,
\]

then there is a unique homomorphism \( \pi \otimes \rho : A \times_{\text{max}} B \to C \) such that

\[
(\pi \otimes \rho)(a \otimes b) = \pi(a)\rho(b) \quad \text{for all } a \in A, b \in B.
\]

Note that every \( C^* \)-norm \( \| \cdot \|_\alpha \) on \( A \otimes B \) is between the maximal and the minimal ones, consequently the identity map on \( A \otimes B \) extends to canonical surjections

\[
A \otimes_{\text{max}} B \to A \otimes_{\alpha} B \to A \otimes_{\text{min}} B.
\]

Some \( C^* \)-algebras \( A \) are sufficiently well-behaved that for every \( C^* \)-algebra \( B \) the maximal and minimal (hence all) \( C^* \)-norms on \( A \otimes B \) coincide. Such an \( A \) is rewarded by being called **nuclear**. Most of the \( C^* \)-algebras we can write down with elementary technology are nuclear — for example: \( K(H) \), every commutative \( C^* \)-algebra, and every finite-dimensional \( C^* \)-algebra. Some nonnuclear ones: \( B(H) \), and (as we'll discuss later in the semester) the \( C^* \)-algebra of the free group on more than one generator.

**Examples.**

(1) If \( (X, \mu) \) and \( (Y, \nu) \) are measure spaces, then the Hilbert space tensor product is

\[
L^2(X, \mu) \otimes L^2(Y, \nu) = L^2(X \times Y, \mu \times \nu).
\]

(2) If \( X \) is a locally compact Hausdorff space with a Radon measure \( \mu \), and \( H \) is a Hilbert space, then the Hilbert space tensor product is

\[
L^2(X) \otimes H = L^2(X, H),
\]

where the RHS is the completion of the pre-Hilbert space \( C_c(X, H) \) with inner product

\[
\langle f, g \rangle = \int_X \langle f(x), g(x) \rangle \, d\mu(x).
\]

(3) If \( X, Y \) are locally compact Hausdorff spaces, then the \( C^* \)-tensor product (unique, since commutative \( C^* \)-algebras are nuclear) is

\[
C_0(X) \otimes C_0(Y) = C_0(X \times Y).
\]

\(^3\)believe it or not
(4) If $X$ is a locally compact Hausdorff space and $A$ is a $C^*$-algebra, then the $C^*$-tensor product is
\[ C_0(X) \otimes A = C_0(X, A). \]
(5) If $M_n$ is a matrix algebra and $A$ is a $C^*$-algebra, then the $C^*$-tensor product (unique, since $M_n$ is finite-dimensional) is
\[ M_n \otimes A = M_n(A), \]
where the RHS comprises the $n \times n$ matrices with entries from $A$.
(6) If $H$ and $K$ are Hilbert spaces, then
\[ \mathcal{K}(H) \otimes \mathcal{K}(K) = \mathcal{K}(H \otimes K). \]

Now work the following problems:

1. If $u$ is a unitary in a unital $C^*$-algebra $A$, prove that there is a unique homomorphism $\pi : C(\mathbb{T}) \to A$ such that $\pi(z) = u$, where $z : \mathbb{T} \to \mathbb{C}$ is the inclusion map. (This one has nothing to do with tensor products, but is needed for the next one.)

2. If $u$ and $v$ are commuting unitaries in $A$, prove that there is a unique homomorphism $\pi : C(\mathbb{T}^2) \to A$ such that, for all $n, k \in \mathbb{Z}$, if $f(z, w) = z^n w^k$ then $\pi(f) = u^n v^k$.

3. If $H$ and $K$ are Hilbert spaces with orthonormal bases $\{e_i\}_{i \in I}$ and $\{f_j\}_{j \in J}$, respectively, then $\{e_i \otimes f_j\}_{i \in I, j \in J}$ is an orthonormal basis of $H \otimes K$. 