Exercise 1. Let $A$ be a unital $C^*$-algebra, and let $a \in A$ be positive, which means that $a$ is self-adjoint and
\[ \sigma(a) \subset [0, \infty). \]

(a) Prove that if $B$ is any commutative $C^*$-subalgebra of $A$ such that $a \in B$ and $1 \in B$ then there is a unique positive element $b \in B$ such that $b^2 = a$. Hint: continuous functional calculus.

(b) Prove that if $b, c \in A$ are positive and $b^2 = c^2 = a$ then $b = c$. Hint: Let $b$ be the unique positive element of $C^*(a, 1)$ such that $b^2 = a$, and prove that $c$ commutes with $a$.

Exercise 2. Let $\mathcal{H}$ be a Hilbert space, and let $U \in B(\mathcal{H})$ be unitary.

(a) Prove that
\[ \sigma(U) \subset \mathbb{T}. \]
Hint: continuous functional calculus.

(b) Prove there exists a self-adjoint operator $T \in B(\mathcal{H})$ such that $e^{iT} = U$. Hint: use the Borel functional calculus with a suitable version of log on $\mathbb{T}$. 