Examples of locally convex spaces

This section gives some standard examples of locally convex spaces defined using seminorms.

1. Let $X$ be a set, and let $\mathbb{C}^X$ be the vector space of functions from $X$ to $\mathbb{C}$ (with the usual pointwise operations). Give $\mathbb{C}^X$ the locally convex topology generated by the seminorms $f \mapsto |f(x)|$ for $x \in X$.

   This is the topology of pointwise convergence, equivalently the product topology.

2. Similarly to the above, let $X$ be a topological space, and let $C(X)$ be the subspace (with the pointwise convergence topology) of $\mathbb{C}^X$ comprising the continuous functions. Actually, this topology is not very useful.

3. Let $X$ be a set, and let $\ell^\infty(X)$ be the vector space of bounded complex-valued functions on $X$ (equivalently, $L^\infty(X,\mu)$, where $\mu$ is the counting measure on $X$). Give $\ell^\infty(X)$ the locally convex topology generated by the norm $\|f\|_\infty = \sup\{|f(x)| : x \in X\}$.

   This is the topology of uniform convergence.

4. Let $X$ be a topological space, and give $C(X)$ the locally convex topology generated by the seminorms $f \mapsto \sup\{|f(x)| : x \in K\}$ for compact $K \subset X$.

   This is the topology of uniform convergence on compact sets, which is quite useful, especially when $X$ is locally compact Hausdorff, when the topology is often called the compact-open topology.

5. Let $U \subset \mathbb{C}$ be open and connected, and let $H(U)$ denote the holomorphic functions on $U$. Then $H(U)$ is a closed subspace of $C(U)$ (with the topology of uniform convergence on compact sets), by the general theory of analytic functions.

6. Some of the above examples have straightforward and useful generalizations where the functions are allowed to take values in a normed space, or even a locally convex space (and of course the scalars can be restricted to the real numbers if convenient). For example, if $X$ is a locally compact Hausdorff space and $V$ is a Banach space, the space $C(X,V)$ with the locally convex topology generated by the seminorms $f \mapsto \sup\{\|f(x)\| : x \in K\}$ for compact $K \subset X$. 
A slight generalization of this one: if $V$ is a locally convex space we can give $C(X, V)$ the locally convex topology generated by the seminorms

$$f \mapsto \sup\{p(f(x)) : x \in K\}$$

for compact $K \subset X$ and $p$ a continuous seminorm on $V$.

7. Let $X, Y$ be Banach spaces. The **strong operator topology** on $B(X, Y)$ is generated by the seminorms

$$T \mapsto \|T(x)\| \quad \text{for } x \in X.$$  

8. With the same notation as the preceding example, the **weak operator topology** is generated by the seminorms

$$T \mapsto |\omega(T(x))| \quad \text{for } x \in X, \omega \in Y^*.$$  

This coincides with the weak topology generated by the linear functionals

$$T \mapsto \omega(T(x)) \quad \text{for } x \in X, \omega \in Y^*,$$

so the weak operator continuous linear functional on $B(X, Y)$ are precisely the linear combinations of the above functionals.

An extremely important special case is where $X = Y$ is a Hilbert space $H$, and then the seminorms generating the weak operator topology on $B(H)$ are

$$T \mapsto |\langle Tx, y \rangle| \quad \text{for } x, y \in H.$$  

9. Let $C^\infty(\mathbb{R}^n)$ be the vector space of infinitely differentiable real-valued functions on $\mathbb{R}^n$. This time we introduce some notation to clean up the definition of the seminorms. For each $i = 1, \ldots, n$ and $f \in C^\infty(\mathbb{R}^n)$ define

$$D_i f = \frac{\partial}{\partial x_i} f,$$

and for any nonnegative integer $k$ let $D_i^k$ be the $k$-th power of the operator $D_i$, so that $D_i f$ is the $k$-th partial derivative of $f$ with respect to $x_i$.

Call an $n$-tuple $(\alpha_1, \ldots, \alpha_n)$ of nonnegative integers a **multi-index**, and for a multi-index $\alpha = (\alpha_1, \ldots, \alpha_n)$ put $|\alpha| = \sum_i \alpha_i$ and define an $|\alpha|$-th order partial differential operator $D^\alpha$ on $C^\infty(\mathbb{R}^n)$ by

$$D^\alpha f = D_1^{\alpha_1} \cdots D_n^{\alpha_n} f.$$  

Now give $C^\infty(\mathbb{R}^n)$ the locally convex topology generated by the seminorms

$$f \mapsto \sup\{|D^\alpha f(x)| : |\alpha| \leq N, \|x\| \leq K\},$$

where $N$ is a nonnegative integer and $K \in \mathbb{N}$.

Since this is a countable family of seminorms, $C^\infty(\mathbb{R}^n)$ is metrizable (because it is the topology induced by a linear embedding into a countable product of normed spaces), and a sequence converges in $C^\infty(\mathbb{R}^n)$ if and only if all of its partial derivatives (of any order, including 0-th order, i.e., the functions themselves) converge uniformly on every compact subset of $\mathbb{R}^n$. Something similar could be done on any open subset $U$ of $\mathbb{R}^n$, after expressing $U$ as a countable union of compact sets $K_j$ such that

$$K_j \subset K_{j+1}^\circ \quad \text{for all } j \in \mathbb{N}.$$
This topology has the feature that every partial differential operator is continuous.

10. Let $U \subset \mathbb{R}^n$ be open. For each compact set $K \subset U$ put

$$C_K(U) = \{ f \in C_c(U) : \text{supp } f \subset K \},$$

and give $C_K(U)$ the uniform norm. Note that the norm topology on $C_K(U)$ is the relative topology as a subspace of $C(U)$ (with the topology of uniform convergence on compact sets). Also note that

$$C_c(U) = \bigcup \{ C_K(U) : K \subset U \text{ compact} \}.$$

However, we do not want to give $C_c(U)$ the relative topology from $C(U)$, because $C_c(U)$ would not be complete in this topology, and completeness is a desirable quality for topological vector spaces. Instead, we introduce the inductive limit topology: for each $K$ let $T_K$ denote the norm topology on $C_K(U)$. There is a strongest locally convex topology on $C_c(U)$ for which a local base comprises all balanced convex sets $V$ such that

$$V \cap C_K(U) \in T_K \quad \text{for all compact } K \subset U.$$

We will not prove this here, because it is a little messy and we do not need inductive limits (at least at this point, and maybe not anytime in this course).

This topology does make $C_c(U)$ complete, and has interesting properties. For example, it is not metrizable (because it is not generated by a countable family of seminorms). Even so, however, a linear map $T$ from $C_c(U)$ into a locally convex space is continuous (for the inductive limit topology on $C_c(U)$) if and only if it is sequentially continuous, that is, $T(f_j) \to T(f)$ whenever the sequence $\{f_j\}$ converges to $f$ in $C_c(U)$. In fact, $T$ is continuous if and only if every restriction $T|C_K(U)$ is.

Something similar can be done involving derivatives of all orders, and the resulting locally convex space $C^\infty_c(U)$ is called the test space on $U$. Elements of the dual space $C^\infty_c(U)^*$ are called distributions on $U$. Test functions and distributions are very important in PDE’s and Fourier analysis.