Compact operators

This section gives the basic properties of compact linear maps.

Throughout this section, $X$, $Y$, and $Z$ will be Banach spaces unless otherwise specified.

**Definition 1.** If $a > 0$ we write $B_a$ for the open ball $\{x : \|x\| < a\}$ in any Banach space, and $B_a(X)$ to emphasize that the Banach space is $X$. If $a = 1$ we just write $B$.

**Definition 2.**

(1) A linear map $T : X \to Y$ is compact if $T(B)$ is relatively compact (that is, has compact closure) in $Y$.

(2) $K(X, Y)$ denotes the set of all compact linear maps from $X$ to $Y$, and $K(X) = K(X, X)$.

**Observation 3.** The following are equivalent:

(1) $T$ is compact;

(2) For every sequence $\{x_n\}$ in $B$, the sequence $\{Tx_n\}$ has a convergent subsequence;

(3) $T(B)$ is totally bounded;

(4) $T$ takes every bounded set to a relatively compact set.

**Observation 4.** Since a relatively compact set is bounded, every compact linear map is bounded.

**Proposition 5.**

(1) $K(X, Y)$ is a closed subspace of $B(X, Y)$.

(2) If $T \in B(X, Y)$ and $S \in B(Y, Z)$, then $ST$ is compact if $T$ or $S$ is.

**Proof.** (1). If $T, S \in K(X, Y)$ and $c \in F$ then

$$(T + S)(B) \subset T(B) + S(B) \quad \text{and} \quad (cT)(B) = cT(B)$$

are totally bounded (easy exercise). Thus $K(X, Y)$ is a subspace.

Now let $\{T_n\}$ be a sequence in $K(X, Y)$ converging to $T \in B(X, Y)$. Let $\varepsilon > 0$. Choose $n$ such that $\|T - T_n\| < \varepsilon/2$. Then choose a finite set $F \subset Y$ such that $T_n(B) \subset F + B_{\varepsilon/2}$. For any $x \in B$ we can choose $y \in F$ such that $\|T_n x - y\| < \varepsilon/2$, and then we have

$$\|Tx - y\| \leq \|Tx - T_n x\| + \|T_n x - y\| < \|T - T_n\|\|x\| + \frac{\varepsilon}{2} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$ 

Thus $T(B) \subset F + B_{\varepsilon}$, and we have shown that $T$ is compact. Therefore $K(X, Y)$ is closed in $B(X, Y)$.

(2). This was an exercise, and depends upon the following fact, which is another easy exercise: every bounded linear map preserves total boundedness. □

**Definition 6.** Let $T : X \to Y$ be linear.
(1) The rank of $T$ is

$$\text{rank } T = \begin{cases} \dim \text{ran } T & \text{if ran } T \text{ is finite-dimensional} \\ \infty & \text{if not.} \end{cases}$$

(2) $T$ is finite rank if $\text{rank } T < \infty$.

Observation 7.

(1) Every finite rank linear map is compact, because of the Heine-Borel Theorem in $\mathbb{R}^n$.

(2) If $T : X \to Y$ is finite rank and $S, R$ are bounded linear maps to $X$ and from $Y$, respectively, then both $TS$ and $RT$ are finite rank.

Definition 8. Let $H$ be a Hilbert space, and let $M$ be a closed subspace. We write $P_M$ for the orthogonal projection onto $M$, i.e., the unique linear operator on $H$ such that

(1) $\text{ran } P_M = M$;

(2) $P_M^2 = P_M$;

(3) $\ker P_M = M^\perp$,

where we recall that in a Hilbert space $M^\perp$ denotes the orthogonal complement of $M$ in $H$.

Observation 9.

(1) With the above notation, $I - P_M$ is the orthogonal projection onto $M^\perp$.

(2) Every orthogonal projection has norm 1 (unless it is the 0 operator, i.e., the orthogonal projection onto the 0 subspace).

Proposition 10. If $H$ is a separable, infinite-dimensional Hilbert space, then $K(H)$ is the closure of the finite rank operators.

Actually, the result is true without the separability assumption, but this special case is all we will need, and it makes the argument a little cleaner.

Proof. Let $T \in K(H)$. Choose an orthonormal basis $\{e_n\}$ for $H$. For each $n$ let $P_n$ denote the orthogonal projection onto $\text{span}\{e_1, \ldots, e_n\}$. Then $P_nT$ is finite rank, and we will show that $P_nT \to T$. Let $Q_n = I - P_n$, which is the orthogonal projection onto $\text{span}\{e_k : k > n\}$. We must show that $\|Q_nT\| \to 0$. Suppose not. Then for all $n$ there exists $x_n \in B$ such that $\|Q_nT x_n\| \geq 1/2$. Pass to a subsequence and relabel so that $\{T x_n\}$ converges, say to $y \in H$. Since $(T x_n - y) \to 0$ and $\{Q_n\}$ is bounded we have $Q_n(T x_n - y) \to 0$. Then since $Q_n y \to 0$ we must have $Q_nT x_n \to 0$, which is a contradiction. □

Proposition 11. If $T \in K(X,Y)$ has closed range, then it is finite rank.

Proof. Replace $Y$ by ran $T$, so that without loss of generality $T$ is surjective. Then $T$ is open by the Open Mapping Theorem. Thus $T(B)$ is a relatively compact neighborhood of 0, so $Y$ is locally compact. But then $Y$ must be finite-dimensional. □

Theorem 12. $T \in B(X,Y)$ is compact if and only if $T^*$ is.
Proof. First assume that $T$ is compact. Then the closure $K$ of $T(B)$ in $Y$ is compact. Let $\{f_n\}$ be a sequence in $B(Y^*)$. Then $\{f_n\}$ is bounded and equicontinuous on $K$ (easy exercise), so by the Arzelà-Ascoli Theorem there is a subsequence $\{g_k\}$ which is uniformly convergent on $K$. Thus $\{T^*g_k\} = \{g_k \circ T\}$ converges uniformly on $B$, so $\{T^*g_k\}$ converges in $X^*$.

Now assume that $T^*$ is compact. Then so is $T^{**}$ be the above, and therefore so is $T = T^{**}|X$ (regarding $X$ and $Y$ as sitting inside the double duals via the canonical embeddings). \qed