Theorem. In a metric space, every convergent sequence is Cauchy, and every Cauchy sequence is bounded.

Theorem. In a metric space, \( t \in \overline{A} \) if and only if there is a sequence in \( A \) converging to \( t \).

Theorem. Let \( X \) be a metric space and \( A \subset X \). Then \( A \) is closed if and only if \( \lim x_n \in A \) whenever \( \{x_n\} \) is a sequence in \( A \) which converges in \( X \).

Theorem. Every closed subset of a complete metric space is complete, and every complete subset of a metric space is closed.

Theorem. In a metric space, \( x \) is a subsequential limit of \( \{x_n\} \) if and only if every neighborhood of \( x \) contains \( x_n \) infinitely often.

Theorem. Let \( X \) and \( Y \) be metric spaces and \( f : X \rightarrow Y \). The following are equivalent:

1. \( f \) is continuous at every point of \( X \);
2. \( f^{-1}(U) \) is open whenever \( U \) is open;
3. \( f(x_n) \rightarrow f(x) \) whenever \( x_n \rightarrow x \).

Theorem. Let \( X \) be a metric space. The following are equivalent:

1. \( X \) is compact;
2. every sequence in \( X \) has a convergent subsequence;
3. \( X \) is complete and totally bounded.

Theorem. Let \( X \) be a metric space and \( A \subset X \). Then \( \overline{A} \) is compact if and only if every sequence in \( A \) has a subsequence which converges in \( X \).

Bolzano-Weierstrass Theorem.

1. Every bounded sequence in \( \mathbb{R}^n \) or \( \mathbb{C}^n \) has a convergent subsequence.
2. Every bounded infinite subset of \( \mathbb{R}^n \) or \( \mathbb{C}^n \) has a cluster point.

Heine-Borel Theorem. A subset of \( \mathbb{R}^n \) or \( \mathbb{C}^n \) is compact if and only if it is closed and bounded.

Theorem. \( \mathbb{R}^n \) and \( \mathbb{C}^n \) are separable.

Theorem. A metric space is separable if and only if it is second countable.
Lindelöf’s Theorem. In a separable metric space, every open cover has a countable subcover.

Theorem. Every compact metric space is separable.

Baire Category Theorem. In a complete metric space, every countable intersection of dense open sets is dense.

Little Baire Theorem. A complete metric space is nonmeager in itself.

Theorem. A continuous function from a compact metric space to a metric space is uniformly continuous.

Theorem. A uniformly Cauchy sequence of functions with values in a complete metric space is uniformly convergent.

Theorem. A uniform limit of continuous functions is continuous.

Theorem. A uniform limit of bounded functions is bounded.

Theorem. Let \( \{f_n\} \) be a uniformly convergent sequence of Riemann integrable functions on \([a, b]\). Then \( f_a^b \lim f_n = \lim f_a^b f_n \).

Theorem. Let \( \{f_n\} \) be a sequence of differentiable functions which converges at some point and whose derivatives converge uniformly. Then \( \{f_n\} \) converges pointwise, and \((\lim f_n)' = \lim f_n'\).

Arzelà-Ascoli Theorem. A sequence of continuous functions from a compact metric space to \( \mathbb{C} \) has a uniformly convergent subsequence if and only if it is equicontinuous and pointwise bounded.

Weierstrass Approximation Theorem. Every continuous function on \([a, b]\) is a uniform limit of polynomials.

Weierstrass M-Test. Let \( \sum f_n \) be a series of bounded functions from \( X \) to \( \mathbb{C} \), and let \( M_n = \sup_X |f_n| \). If \( \sum M_n \) converges, then \( \sum f_n \) converges uniformly.

Cauchy-Hadamard Theorem. The radius of convergence of a power series \( \sum c_n(x-a)^n \) is \( 1/\limsup |c_n|^{1/n} \).

Theorem. A power series \( \sum c_n(x-a)^n \) with radius of convergence \( r \) can be differentiated and integrated term-by-term in the interval \((a-r, a+r)\).

Taylor Series. If \( f(x) = \sum c_n(x-a)^n \) on an open interval containing \( a \), then \( c_n = f^{(n)}(a)/n! \).