Unless otherwise specified, all vector spaces have scalar field $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$.

**Theorem.** For a subset $B$ of a vector space $X$, the following are equivalent:

1. $B$ is a basis;
2. $B$ is linearly independent and spans $X$;
3. every vector in $X$ can be written uniquely as a linear combination of vectors in $B$.

**Theorem.** Every linearly independent subset of a vector space $X$ can be extended to a basis of $X$.

**Theorem.** Every spanning subset of a vector space $X$ contains a basis of $X$.

**Theorem.** Let $X$ and $Y$ be vector spaces, $\{u_i\}_{i \in I}$ a basis of $X$, and $\{v_i\}_{i \in I} \subset Y$. Then there exists a unique linear map $T: X \to Y$ such that $Tu_i = v_i$ for all $i \in I$.

**Theorem.** Let $X$ be a vector space with basis $B = \{u_i\}_{i \in I}$. For each $i \in I$ let $f_i$ be the unique linear functional on $X$ such that

$$f_i(u_j) = \begin{cases} 1 & \text{if } j = i \\ 0 & \text{if not.} \end{cases}$$

Then:

1. for every $x \in X$, $x = \sum_{i \in I} f_i(x)u_i$ is the expression of $x$ as a linear combination of the basis vectors;
2. the map $f = (f_i)_{i \in I}$ gives an isomorphism of $X$ onto the direct sum $\bigoplus_{i \in I} \mathbb{F}$, where $\mathbb{F}$ is the scalar field. In particular, if $I = \{1, \ldots, n\}$ then $f$ is an isomorphism of $X$ onto $\mathbb{F}^n$.

**Theorem.** Let $X$ and $Y$ be finite-dimensional vector spaces with bases $B = \{u_1, \ldots, u_n\}$ and $C = \{v_1, \ldots, v_m\}$, respectively. For each $i = 1, \ldots, n, j = 1, \ldots, m$ let $E_{ij}: X \to Y$ be the unique linear map such that

$$E_{ij}u_k = \begin{cases} v_i & \text{if } k = j \\ 0 & \text{if not.} \end{cases}$$

Then:

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1. \( \{E_{ij} : i = 1, \ldots, n, j = 1, \ldots, m\} \) is a basis for the vector space of all linear maps from \( X \) to \( Y \);

2. for each linear map \( T : X \to Y \), let \( T = \sum_{ij} a_{ij}E_{ij} \). Then the \( m \times n \) matrix \( A = (a_{ij}) \) has the following property: if \( x \in X \) has \( n \)-tuple \( (x_1, \ldots, x_n) \) relative to the basis \( B \), then the \( i \)-th coordinate of the \( m \)-tuple of \( Tx \) relative to the basis \( C \) is given by

\[
(Tx)_i = \sum_{j=1}^{n} a_{ij}x_j.
\]

**Theorem.** Let \( B \) and \( C \) be bases for vector spaces \( X \) and \( Y \), respectively. Then \( X \) and \( Y \) are isomorphic if and only if \( B \) and \( C \) have the same cardinality. In particular, if \( X \) and \( Y \) are finite-dimensional, then \( X \cong Y \) if and only if \( \dim X = \dim Y \).

**Theorem.** Let \( T : X \to Y \) be linear, let \( Z \) be a subspace of \( X \), and let \( Q : X \to X/Z \) be the quotient map. Then there exists a linear map \( S : X/Z \to Y \) making the diagram

\[
\begin{array}{ccc}
X & \xrightarrow{T} & Y \\
\downarrow{Q} & & \nearrow{S} \\
X/Z & & \\
\end{array}
\]

commute, i.e., such that \( T = S \circ Q \), if and only if \( \ker T \supseteq Z \), in which case \( \ker S = (\ker T)/Z \).

**Theorem.** For every subspace \( Y \) of a vector space \( X \) there is a complementary subspace, i.e., a subspace \( Z \) such that \( X = Y \oplus Z \).

**Theorem.** If \( X = Y \oplus Z \), then there is a unique idempotent linear operator \( P \) on \( X \) with range \( Y \) and kernel \( Z \). Conversely, if \( P \) is an idempotent linear operator on \( X \), then \( X = \text{ran} T \oplus \ker T \).

**Theorem.** If \( T : X \to Y \) is linear and \( X \) is finite-dimensional, then

\[
\dim X = \dim \text{ran} T + \dim \ker T.
\]

**Theorem.** Let \( f \) and \( g_1, \ldots, g_n \) be linear functionals on a vector space \( X \). Then \( f \) is a linear combination of \( g_1, \ldots, g_n \) if and only if \( \ker f \supseteq \bigcap_{i=1}^{n} \ker g_i \).

**Theorem.** If \( Y \) and \( Z \) are subspaces of a vector space \( X \), the assignment \( (y, z) \mapsto y + z \) gives a linear map of the external direct sum \( Y \oplus Z \) onto the sum \( Y + Z \), with kernel \( \{(x, -x) : x \in Y \cap Z\} \). In particular, if \( X \) is finite-dimensional then

\[
\dim(Y + Z) = \dim Y + \dim Z - \dim(Y \cap Z).
\]

**Cauchy-Schwartz Inequality.** Let \( X \) be an inner product space and \( x, y \in X \). Then

\[
|\langle x, y \rangle|^2 \leq \langle x, x \rangle \langle y, y \rangle.
\]
Theorem. If $Y$ is a subspace of a finite-dimensional inner product space $X$, then:

1. $Y^\perp \perp = Y$,
2. $X = Y \oplus Y^\perp$, and
3. the unique idempotent operator $P$ on $X$ with range $Y$ and kernel $Y^\perp$ is self-adjoint (i.e., $P = P^*$).

Real Spectral Theorem. If $T$ is a linear operator on a finite-dimensional real inner product space $X$, then $T$ is self-adjoint if and only if $X$ has an orthonormal basis of eigenvectors of $T$.

Complex Spectral Theorem. If $T$ is a linear operator on a finite-dimensional complex inner product space $X$, then $T$ is normal (i.e., $TT^* = T^*T$) if and only if $X$ has an orthonormal basis of eigenvectors of $T$. 