INTEGRATION RESULTS

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**Theorem.** The set of measurable functions on a measurable space is a vector space which is closed under pointwise sequential limits.

**Theorem.** The set of integrable functions on a measure space is a vector space on which the integral is a linear functional.

**Theorem.** On every $\mathbb{R}^n$, Lebesgue measure, and the $\sigma$-algebra of Lebesgue measurable sets, exist, are unique, and are complete and $\sigma$-finite.

**Theorem.** On a measurable space, if $f$ is measurable, then there exists a sequence $\{\phi_n\}$ of simple functions such that $|\phi_n| \leq |f|$ and $\phi_n \to f$ pointwise. Moreover, if $f$ is nonnegative then without loss of generality $0 \leq \phi_n \nearrow f$.

**Theorem.** On a measure space $(X, \mu)$, if $f$ is measurable, then $f$ is integrable if and only if $|f|$ is, in which case $$\left| \int f \, d\mu \right| \leq \int |f| \, d\mu.$$ 

**Monotone Convergence Theorem.** If $\{f_n\}$ is an increasing sequence of nonnegative measurable functions, then $$\lim_{n \to \infty} \int f_n \, d\mu = \int \lim_{n \to \infty} f_n \, d\mu.$$ 

**Fatou's Lemma.** If $\{f_n\}$ is a sequence of nonnegative measurable functions, then $$\liminf \int f_n \, d\mu \leq \int \liminf f_n \, d\mu.$$ 

**Dominated Convergence Theorem.** Let $\{f_n\}$ be an a.e.-convergent sequence of integrable functions. If there exists an integrable function $g$ such that $|f_n| \leq g$ a.e. for all $n$, then $\lim f_n$ is integrable and $$\int \lim f_n \, d\mu = \lim \int f_n \, d\mu.$$ 

**Series Version of DCT.** If $\sum_n f_n$ is a series of integrable functions such that $\sum_n |f_n| < \infty$ a.e., then $\sum_n f_n$ converges a.e., the sum is integrable, and $$\int \sum_n f_n \, d\mu = \sum_n \int f_n \, d\mu.$$ 

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Theorem. If $f$ is Lebesgue integrable on $\mathbb{R}^n$, then there exists a sequence $\{g_n\}$ of continuous functions with compact support such that:

1. $|g_n| \leq \sup |f|$ for all $n$;
2. $g_n \to f$ a.e.;
3. $\int |f - g_n| \to 0$.

Theorem. A bounded function $f: [a, b] \to \mathbb{R}$ is Riemann integrable if and only if it is continuous a.e., in which case it is also Lebesgue integrable and the two integrals agree.

Fubini’s Theorem. Let $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$ be complete sigma-finite measure spaces. If $f$ is $\mu \times \nu$-integrable on $X \times Y$, then:

1. $f(x, y)$ is a $\mu$-integrable function of $x$ for $\nu$-a.e. $y \in Y$;
2. the $\nu$-a.e.-defined function $y \mapsto \int f(x, y) \, d\mu(x)$ on $Y$ is $\nu$-integrable;
3. $\iint f(x, y) \, d\mu(x) \, d\nu(y) = \int f \, d(\mu \times \nu)$.

Tonelli’s Theorem is similar, but replaces “integrable” by “nonnegative and measurable”.

Theorem. Lebesgue measure on $\mathbb{R}^{n+m}$ is the completion of the product of the Lebesgue measures on $\mathbb{R}^n$ and $\mathbb{R}^m$.

Change of Variables Theorem. Let $\phi$ be a $C^1$ diffeomorphism of an open set $U \subset \mathbb{R}^n$ onto an open set $\phi(U) \subset \mathbb{R}^n$, and let $f$ be an integrable function on $\phi(U)$. Then

$$\int_{\phi(U)} f(u) \, du = \int_U f(\phi(x)) \, |\det \phi'(x)| \, dx.$$