LINEAR ALGEBRA DEFINITIONS

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Unless otherwise specified, all vector spaces have scalar field \( F = \mathbb{R} \) or \( \mathbb{C} \).

A (Hamel) basis for a vector space is a maximal linearly independent subset.

Let \( X \) and \( Y \) be vector spaces and \( T: X \to Y \). Then \( T \) is linear if for all \( x, y \in X \) and \( c \in F \) we have \( T(x + y) = Tx + Ty \) and \( T(cx) = cTx \).

A linear functional on a vector space \( X \) is a linear map from \( X \) to \( F \).

Let \( X \) be a vector space, \( A, B \subset X \), \( z \in X \), \( C \subset F \), and \( d \in F \). Then:

\[
\begin{align*}
A + B &:= \{x + y : x \in A, y \in B\} \\
z + A &:= \{z\} + A \\
CA &:= \{cx : c \in C, x \in A\} \\
dA &:= \{d\}A \\
Cz &:= C\{x\}.
\end{align*}
\]

More generally, if \( \{A_i\}_{i \in I} \) is a family of subsets of \( X \), then

\[
\sum_{i \in I} A_i := \left\{ \sum_{i \in I} x_i : x_i \in A_i \text{ for all } i \in I \text{ and } x_i \text{ is only finitely nonzero} \right\}.
\]

Let \( Z \) be a subspace of a vector space \( X \). The quotient space \( X \) modulo \( Z \) is the set \( X/Z = \{x + Z : x \in X\} \), with operations

\[
(x + Z) + (y + Z) = (x + y) + Z \quad \text{and} \quad c(x + Z) = (cx) + Z
\]

for \( x, y \in X \), \( c \in F \).

Moreover, the function \( Q: X \to X/Z \) defined by \( Qx = x + Z \) is the quotient map.

Let \( T: X \to Y \) be linear.

1. The range of \( T \) is \( \text{ran } T := T(X) = \{Tx : x \in X\} \).

2. The kernel of \( T \) is \( \ker T := T^{-1}(\{0\}) = \{x \in X : Tx = 0\} \).
Let \( \{X_i\}_{i \in I} \) be a family of vector spaces.

1. The direct product is the cartesian product \( \prod_{i \in I} X_i \) with the operations \((x_i) + (y_i) = (x_i + y_i)\) and \(c(x_i) = (cx_i)\).

2. The direct sum is

\[
\bigoplus_{i \in I} X_i := \left\{ (x_i) \in \prod_{i \in I} X_i : x_i = 0 \text{ for all but finitely many } i \in I \right\}
\]

A family \( \{Y_i\}_{i \in I} \) of subspaces of a vector space is independent if for all finite \( J \subset I \) and \( x_i \in Y_i \) for \( i \in J \), if \( \sum_{i \in J} x_i = 0 \) then \( x_i = 0 \) for all \( i \in J \).

If \( \{Y_i\}_{i \in I} \) is an independent family of subspaces of a vector space, then \( \sum_{i \in I} Y_i \) is the internal direct sum of the \( Y_i \)'s, and \( \bigoplus_{i \in I} Y_i \) is the external direct sum of the \( Y_i \)'s. The map

\[
(x_i) \mapsto \sum_{i \in I} x_i : \bigoplus_{i \in I} Y_i \to \sum_{i \in I} Y_i
\]

is an isomorphism, and we blur the distinction between the terminology and notation for the external and internal direct sums.

Let \( Y \) and \( Z \) be subspaces of a vector space \( X \). \( Z \) is an algebraic complement of \( Y \) if \( Y \) and \( Z \) are independent.

A linear operator \( T \) on \( X \) is idempotent if \( T^2 = T \).