Theorem. If $X$ is a 1st countable topological vector space, then:

1. there is a compatible invariant metric for which the balls centered at 0 are balanced, and
2. if $X$ locally convex then the metric can be chosen so that the balls are convex.

Thus, a topological vector space is metrizable as a topological space if and only if it has a compatible invariant metric.

A sequence $\{x_n\}$ in a topological vector space is Cauchy if for every neighborhood $V$ of 0 there exists $k \in \mathbb{N}$ such that $x_n - x_j \in V$ for all $n, j \geq k$.

Proposition. In a metrizable topological vector space, a sequence is Cauchy if and only if it is Cauchy with respect to some compatible invariant metric. In particular, all compatible invariant metrics have the same Cauchy sequences, so either all are complete or none are.

Theorem. Let $Y$ be a subspace of a topological vector space $X$. If $Y$ is an $F$-space, then it is closed.

Lemma. In a topological vector space with a compatible invariant metric $d$, we have $d(nx, 0) \leq nd(x, 0)$ for all $n \in \mathbb{N}$.

Lemma. In a metrizable topological vector space, if $x_n \to 0$ then there exist scalars $c_n \to \infty$ such that $c_n x_n \to 0$. 

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