Throughout this discussion, $H$ will be a Hilbert space over $\mathbb{C}$.

**Functional Calculus.** Let $T$ be a bounded normal operator on $H$, and let $A$ be the unital $C^*$-subalgebra of $B(H)$ generated by $T$. Then there is a unique isometric $*$-isomorphism

$$f \mapsto f(T): C(\sigma(T)) \xrightarrow{\cong} A$$

which takes 1 to 1 and $z$ to $T$.

More precisely, by “$z$” in the above we mean the function $g \in C(\sigma(T))$ defined by $g(z) = z$.

**Proof.** Let $\Delta$ be the maximal ideal space of the commutative unital $C^*$-algebra $A$, and let

$$S \mapsto \hat{S}: A \rightarrow C(\Delta)$$

be the Gelfand transform, which is an isometric $*$-isomorphism by the Gelfand-Naimark Theorem. Let $\phi: C(\Delta) \rightarrow A$ be the inverse of the Gelfand transform. Note that $\phi(\hat{T}) = T$ and $\phi(1) = 1$.

Note that $\hat{T}$ is a continuous function of $\Delta$ onto $\sigma(T)$. Now, the polynomials in $T$ and $T^*$ are dense in $A$, so the polynomials in $\hat{T}$ and $\overline{T}$ are dense in $C(\Delta)$. Since the complex homomorphisms of $A$ preserve adjoints, the function $\hat{T}$ separates the points of $\Delta$. Thus $\hat{T}$ is a homeomorphism of $\Delta$ onto $\sigma(T)$.

Hence the map $f \mapsto f \circ \hat{T}$ is an isometric $*$-isomorphism of $C(\sigma(T))$ onto $C(\Delta)$. Note that this isomorphism takes $z$ to $\hat{T}$ and 1 to 1.

Thus the composition $f \mapsto \phi(f \circ \hat{T})$ is an isometric $*$-isomorphism of $C(\sigma(T))$ onto $A$.

For the uniqueness, note that the polynomials in $z$ and $\overline{z}$ are dense in $C(\sigma(T))$.  

QED